Asymptotics of the one dimensional forest-fire processes

Jean-Maxime Le Cousin

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Présentée par

Jean-Maxime Le Cousin

Asymptotique des feux rares dans le modèle des feux de forêts

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Directeur de thèse : Nicolas FOURNIER

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M. Xavier BRESSAUD	Université Paul Sabatier	Examinateur
M. Nathanaël ENRIQUEZ	Université Paris 10	Examinateur
M. Nicolas FOURNIER	Université Paris 6	Directeur de thèse
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M. Arnaud LE NY	Université Paris-Est Créteil	Examinateur
M. Christophe SABOT	Université Lyon 1	Rapporteur

Thèse préparée au
LAMA CNRS UMR 8050
Université Paris-Est Créteil Val de Marne
61, avenue du Général de Gaulle
94010 Créteil cedex, France

Résumé

Dans cette thèse, nous nous intéressons à deux modèles de feux de forêts définis sur $\mathbb{Z}$.

On étudie le *modèle des feux de forêts sur* $\mathbb{Z}$ *avec propagation non instantanée* dans le chapitre 2. Dans ce modèle, chaque site a trois états possibles : *vide*, *occupé* ou *en feu*. Un site vide devient occupé avec taux 1. Sur chaque site, des allumettes tombent avec taux λ. Si le site est occupé, il brûle pendant un temps exponentiel de paramètre π avant de se propager à ses deux voisins. S'ils sont eux-mêmes occupés, ils brûlent, sinon le feu s'éteint. On étudie l'asymptotique des feux rares c'est à dire la limite du processus lorsque $\lambda \to 0$ et $\pi \to \infty$. On montre qu'il y a trois catégories possibles de limites d'échelles, selon le régime dans lequel λ tend vers 0 et π vers l'infini.

On étudie formellement et brièvement dans le chapitre 3 le *modèle des feux de forêts sur* $\mathbb{Z}$ *en environnement aléatoire*. Dans ce modèle, chaque site n'a que deux états possibles : *vide* ou *occupé*. On se donne un paramètre $\lambda > 0$, une loi ν sur $(0\,,\infty)$ et une suite $(\kappa_i)_{i\in\mathbb{Z}}$ de variables aléatoires indépendantes identiquement distribuées selon ν. Un site vide i devient occupé avec taux κ_i. Sur chaque site, des allumettes tombent avec taux λ et détruisent immédiatement la composante de sites occupés correspondante. On étudie l'asymptotique des feux rares. Sous une hypothèse raisonnable sur ν, on espère que le processus converge, avec une renormalisation correcte, vers un modèle limite. On s'attend à distinguer trois processus limites différents.

Mots clés : Systèmes de particules en interaction, Criticalité auto-organisée, Modèles de feux de forêts

Abstract

The aim of this work is to study two differents *forest-fire processes defined on* $\mathbb{Z}$.

In Chapter 2, we study the so-called *one dimensional forest-fire process with non instantaeous propagation*. In this model, each site has three possible states: 'vacant', 'occupied' or 'burning'. Vacant sites become occupied at rate 1. At each site, ignition (by lightning) occurs at rate λ. When a site is ignited, a fire starts and propagates to neighbors at rate π. We study the asymptotic behavior of this process as $\lambda \to 0$ and $\pi \to \infty$. We show that there are three possible classes of scaling limits, according to the regime in which $\lambda \to 0$ and $\pi \to \infty$.

In Chapter 3, we study formally and briefly the so-called *one dimensional forest-fire processes in random media*. Here, each site has only two possible states: 'vacant' or 'occupied'. Consider a parameter $\lambda > 0$, a probability distribution ν on $(0\,,\infty)$ as well as $(\kappa_i)_{i\in\mathbb{Z}}$ an i.i.d. sequence of random variables with law ν. A vacant site i becomes occupied at rate κ_i. At each site, ignition (by lightning) occurs at rate λ. When a site is ignited, the fire destroys the corresponding component of occupied sites. We study the asymptotic behavior of this process as $\lambda \to 0$. Under some quite reasonable assumptions on the law ν, we hope that the process converges, with a correct normalization, to a limit forest fire model. We expect that there are three possible classes of scaling limits.

Key words: Stochastic interacting particle systems, Self-organized criticality, Forest fire model

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I.1. Motivations de la physique statistique

I.1.1. L'ubiquité des fractales et des lois puissances

L'étude et l'analyse des phénomènes qui nous entourent ont été simplifiées et approfondies par l'évolution de la puissance de calculs (traitement de volumineuses bases de données). Ces observations ont fait émerger des phénomènes de *lois puissances*, c'est-à-dire des phénomènes dont la probabilité d'observer des valeurs extrêmement grandes n'est pas exponentiellement bornée (*queue de distribution lourde*). Les phénomènes en lois puissances sont d'une apparente ubiquité dans la nature et sont l'empreinte des structures *fractales* (des structures dont aucune échelle ne prédomine). Ils sont par exemple observés dans

- les tremblements de terre : la loi de Gutenberg-Richter énonce que la probabilité d'obtenir un tremblement de terre d'énergie E est de l'ordre de E^{-B}, où l'exposant B varie dans l'intervalle $[0.80\,, 1.05]$ (en fonction de la précision des mesures) ;

- le littoral : c'est l'exemple classique de fractale dans la nature, devenu populaire dans le contexte de chaos et des fractales ([Man82]). La longueur de la côte de Grande-Bretagne croît de façon non linéaire (vers l'infini) lorsque le pas de mesure tend vers 0 (plus on cherche de précision, plus la longueur augmente, et ce de façon non linéaire). La dimension de Hausdorff est de 1,24 ;

- les marchés financiers : l'évolution du cours d'une action sur une décennie a la *même allure* que sur une année.

On pourra consulter l'ouvrage de P. Bak [Bak96] pour retrouver ces exemples et bien d'autres encore.

Les structures fractales se retrouvent dans les *phénomènes critiques*. Afin d'illustrer la notion de phénomène critique, intéressons-nous à un modèle bien connu des mathématiciens : la percolation.

I.1.2. La percolation : un exemple de phénomène critique

Soit $G = (S, A)$ un graphe, où S est l'ensemble des sommets et A l'ensemble des arêtes. La percolation par arêtes sur le graphe G est un modèle de graphes aléatoires obtenu en supprimant chaque arête de G avec probabilité $1 - p$, indépendamment des autres arêtes (on dit qu'on ferme les arêtes avec probabilité $1 - p$). On dit que x mène à y s'il existe un chemin d'arêtes ouvertes qui va de x à y. Bien que la définition du modèle soit élémentaire, celui-ci exhibe un comportement très riche. La théorie de la percolation étudie la géométrie des composantes connexes du sous-graphe aléatoire induit par les arêtes ouvertes. Intéressons-nous au réseau $\mathbb{Z}^d$ ($d \geq 1$), c'est-à-dire au graphe dont l'ensemble des sommets est $\mathbb{Z}^d$, muni de l'ensemble des arêtes canoniquement associé (deux sommets sont reliés si et seulement si ils sont à une distance 1). Dans la suite, on note $\mathcal{C}$ la composante connexe contenant l'origine. Un résultat fondamental de la percolation (voir par exemple [Gri99]) énonce qu'il existe un nombre $p_c = p_c(d) \in (0\,,1)$, tel que si $p < p_c$, alors il n'y a presque sûrement pas de composante connexe infinie tandis que si $p > p_c$, alors il existe presque sûrement une unique composante connexe infinie. On dit alors qu'il y a une *transition de phase* au *point critique* $p = p_c$. La valeur critique sépare la *phase sous-critique*, lorsque $p < p_c$, où toutes les composantes connexes sont presque sûrement finies, de la *phase sur-critique*, lorsque $p > p_c$, où il y a presque sûrement une composante connexe infinie. Sur $\mathbb{Z}$, on a (trivialement) $p_c = 1$ et sur $\mathbb{Z}^2$ on a $p_c = 1/2$.

Les phases sous- et sur-critiques de la percolation par arêtes sur le réseau $\mathbb{Z}^d$ ($d \geq 2$), est plus ou moins connue. Par exemple, en notant $\mathbb{P}_p$ la mesure gouvernant la configuration, on peut trouver dans [Gri99] les estimations suivantes :

$$\mathbb{P}_p[|\mathcal{C}| \geq k] \leq C_1(p)\exp(-c_1(p)k) \text{ si } p < p_c,$$
$$\mathbb{P}_p[k \leq |\mathcal{C}| < \infty] \leq C_2(p)\exp(-c_2(p)k^{(d-1)/d}) \text{ si } p > p_c,$$
$$\mathbb{P}_p[x \in \mathcal{C}, |\mathcal{C}| < \infty] \leq \exp(-c_3(p)|x|) \text{ si } p \neq p_c.$$

Bien que ces résultats ne soient déjà pas faciles à établir, la percolation au point critique $p = p_c$ est encore mal comprise et des comportements bien différents sont observés (par simulation numérique). Par exemple, l'amas $\mathcal{C}$ est-il $\mathbb{P}_{p_c}$ fini ? En dimension 2, pour des raisons de symétries (et dualité), il est facile de voir (heuristiquement en tout cas) que la composante $\mathcal{C}$ est $\mathbb{P}_{p_c}-$presque sûrement finie. Qu'en est il dans le cas général ? Quel est le comportement de $\mathcal{C}$ au *voisinage* du point critique ?

Pour la percolation par site (on ouvre chaque site avec probabilité p) sur le réseau triangulaire, on a $p_c = 1/2$ et, en combinant les résultats [Kes87], [LSW02], [Smi01] et [SW01], on peut montrer que

$$\begin{aligned}
\mathbb{P}_p[|\mathcal{C}| = \infty] &= (p - 1/2)^{5/36+o(1)} \text{ quand } p \downarrow p_c, \\
\mathbb{P}_{p_c}[|\mathcal{C}| \geq k] &= k^{-5/91+o(1)} \text{ quand } k \to \infty, \\
\mathbb{E}_p[|\mathcal{C}|\, \mathbf{1}_{\{|\mathcal{C}|<\infty\}}] &= |p - p_c|^{-43/18+o(1)} \text{ quand } p \to p_c, \\
\mathbb{P}_{p_c}[\mathrm{rad}|\mathcal{C}| \geq n] &= n^{-5/48+o(1)} \text{ quand } n \to \infty, \\
\mathbb{P}_{p_c}[x \in \mathcal{C}] &= \frac{1}{|x|^{5/24+o(1)}} \text{ quand } |x| \to \infty.
\end{aligned}$$

où $\mathrm{rad}(|\mathcal{C}|) = \sup\{|x| : x \in \mathcal{C}\}$.

En physique, les phénomènes de transitions de phases sont très complexes et la transition de phase est très difficile à expliquer du point de vue microscopique. Même les modèles les plus simples sont difficiles à étudier au point critique. Or dans la nature, ce sont bien souvent des phénomènes critiques (invariances d'échelles, structures fractales, lois puissances...) que l'on observe. Des mécanismes aussi simples que le modèle de percolation ne peuvent expliquer à eux seuls l'apparition de tels phénomènes, car, pour les observer, il faut régler finement le paramètre du modèle sur le point critique. Existe-t-il un modèle simple et universel, c'est-à-dire qui décrive une large classe de phénomènes, qui puisse expliquer l'apparition des fractales et des lois puissances ?

Pour tenter d'expliquer l'apparition ces phénomènes, les physiciens Per Bak, Chao Tang et Kurt Wiesenfeld ont introduit [BTW87] le concept de *criticalité auto-organisée* (SOC : *Self-Organized Criticality*).

I.1.3. Le concept de criticalité auto-organisée

Un système hors d'équilibre (ouvert sur l'extérieur), régi par des interactions locales (*microscopiques*), évolue de lui même (*auto-organisation*) jusqu'à un état critique (dans le sens où il n'y a pas de grandeur caractéristique dominante), à partir duquel une réorganisation locale peut avoir des répercussions globales (influencer une partie macroscopique du système) : c'est ainsi que peut être définie la notion de criticalité auto-organisée. Dans cette théorie, l'évolution du système vers un état critique est déterminée par des règles locales et non pas par un expérimentateur qui réglerait un paramètre (dans le modèle de percolation, on peut décider d'augmenter le paramètre s'il n'y a que des amas finis et de le baisser s'il y a un amas infini ; ce modèle se développe trivialement vers un état critique). Ce concept peut être décrit comme suit : considérons un système de particules en interactions, avec une configuration initiale quelconque, qui est régi par des interactions locales et telle que l'évolution des forces extérieures soit lente (il y a une séparation entre les échelles de temps du processus interne et celui du processus externe). Un tel système devrait évoluer naturellement vers un état critique sans le réglage de paramètres extérieurs. L'état critique est un état instable mais stationnaire qui doit présenter les caractéristiques suivantes :

- les interactions entre les sites sont locales ;
- il y a un effet de seuil : on observe une activité (macroscopique) seulement si un certain seuil est atteint ;
- il y a un effet dissipatif, pour compenser l'évolution des paramètres externes ;
- les distributions des observations sont en lois puissances.

Pour expliquer ce concept, les auteurs [BTW87] ont proposé le modèle de tas de sable suivant : considérons une table plate sur laquelle des grains de sables tombent, lentement, un par un. Les grains peuvent être ajoutés aléatoirement sur la table ou sur un endroit particulier, le centre de la table par exemple. L'état où tous les grains sont au même niveau est un état d'équilibre. Comme les grains ont tendance à s'immobiliser du fait de la friction, on ne revient pas automatiquement à l'état plat quand on arrête d'ajouter des grains. Au début, les grains de sable restent plus ou moins à l'endroit ou ils tombent. Si on continue d'en ajouter, le tas devient plus gros et des grains de sable glissent ou créent des avalanches. Les grains peuvent atterrir sur d'autres grains ou glisser plus bas dans le tas. Cela peut créer d'autres glissements de grains. L'ajout d'un simple grain peut causer des turbulences locales mais le tas reste stable dans son ensemble. En particulier, les évènements dans une partie du tas n'affectent pas les grains de sable situés plus loin : il n'y a pas de communication globale dans le tas, juste des grains de sable seuls.

Plus la pente augmente, plus l'ajout d'un simple grain est susceptible de créer des glissements d'autres grains. Finalement, la pente atteint une certaine valeur et ne peut plus augmenter, car la quantité de sable ajoutée compense en moyenne la quantité de sable qui tombe de la table. On appelle cela un *état stationnaire* car la quantité de sable et la pente sont constantes en moyenne au cours du temps. Il est alors clair que pour avoir cette compensation entre l'ajout de sable au centre de la table et la perte de sable sur les bords de la table, il doit y avoir une communication à travers toute la pile. On appelle cette configuration l'*état critique auto-organisé*.

L'ajout de grains de sable a transformé le système d'une configuration où les grains de sable suivent leurs propre dynamique en un état critique où les dynamiques sont globales. Dans l'état stationnaire SOC, il y a *un* système complexe avec sa propre dynamique. On ne pouvait pas prévoir *a priori* l'émergence du tas des propriétés individuelles des grains.

Le tas de sable est un système dynamique ouvert car les grains sont ajoutés de l'extérieur. L'état critique doit être consistant : c'est important pour avoir une chance que le modèle décrive le monde réel.

Les tailles des avalanches peuvent être mesurées de plusieurs façon, par exemple en étudiant la durée d'une avalanche ou le nombre de sites affectés. On espère que toutes ces quantités présentent des distributions en lois puissances. Malgré son nom, le modèle du tas de sable n'a pas été introduit pour décrire les tas de sables réels mais pour expliquer abstraitement l'émergence des systèmes critiques auto-organisés. En particulier, les auteurs de [BTW87] envisageaient leur modèle comme une justification abstraite de l'omniprésence des réponses en $1/f$.

I.1.4. Les limites des SOC

La théorie développée dans [BTW87] est fascinante car elle offre une explication simple et universelle à des phénomènes très divers qui, comme les tas de sable, semblent obéir statistiquement à des lois de puissance : l'évolution des cours de bourse, les statistiques sur la taille des villes, l'audience des sites sur internet, la fréquence des mots, le nombre d'espèces par genre, l'intensité des guerres et des éruptions solaires, l'importance des catastrophes géologiques.

Malheureusement, l'observation de lois puissances ne suffit pas à décider si un système est SOC ou non. De plus, il est assez difficile de décider si une distribution suit une loi puissance ou non : faute de mesures faites sur plusieurs ordres de grandeur, on se laisse abuser par une loi géométrique ou exponentielle. En 2009 Clauset et ses co-auteurs [CNS09] ont discuté de la pertinence de la distribution en lois puissances de 26 modèles : seulement deux modèles ont été validés. Les mesures sur les systèmes biologiques s'étalent sur un trop petit nombre d'années pour être significatives. De plus, même si une distribution en loi puissance est détectée, rien ne dit que l'on a affaire à un système critique auto-organisé.

I.2. Introduction du modèle de feux de forêts

Dans cette partie, on introduit un modèle qui, sous certaines conditions, doit exhiber un comportement SOC : le modèle des feux de forêts (MFF).

Une première version du MFF a été proposée par Bak, Chen et Tang [BCT90] mais la criticalité a été très rapidement invalidée [GK91] (par simulations numériques). Le modèle qui suit est connue sous le nom de modèle des feux de forêts critique de Drossel-Schwabl (DS-MFF) et a été introduit par Barbara Drossel et Hantz Schwabl [DS92]. Il est intimement lié à la percolation par sites et hérite de certaines de ses notations et propriétés (technique de simulation, exposants critiques).

Soit d un entier naturel. Considérons $B \coloneqq [-L\,,L]^d \cap \mathbb{Z}^d$ avec L *assez grand*. Le DS-MFF est un processus de Markov à temps discret sur B dans lequel chaque site peut être soit *occupé* (par un arbre), soit en *feu* (occupé par un arbre en feu) ou soit *vide* (en cendre). Le processus part d'une configuration initiale quelconque. À chaque étape, la configuration change suivant les règles suivantes :

- chaque site libre devient occupé par un arbre avec probabilité p ;
- chaque site en feu devient vide ;
- si un site était occupé et avait un de ses voisins en feu, il devient en feu ;

- si un site était occupé et n'avait aucun de ses voisins n'est en feu, il devient en feu avec probabilité f.

Ce modèle contient deux échelles de temps : p^{-1}, qui représente la fréquence d'apparition des gaines, et f^{-1}, qui représente la fréquence d'apparition des feux. Pour que le système se développe en système SOC, l'échelle de temps du mécanisme externe (apparition des graines et des feux) doit être beaucoup plus grande que celle du mécanisme interne (propagation des feux). Ainsi, pour espérer observer de la criticalité, il est raisonnable d'imposer

$$p, f \to 0. \tag{1}$$

La propagation des feux (troisième étape) est alors *instantanée* : comme nous sommes dans une boîte finie et que les taux d'apparition des arbres et des feux sont très petits, le feu se sera propagé avant qu'un arbre ou qu'un feu n'apparaisse à nouveau. Autrement dit, entre deux étapes, si une allumette tombe dans un amas, il le détruit alors instantanément. Malheureusement, il a été montré (par simulations) que cela ne suffit pas pour développer de la criticalité. Dans l'état stationnaire, l'afflux d'arbres doit en compenser la perte, et donc la relation

$$p\rho_e = f\rho_o\langle s\rangle$$

doit être vérifiée, où ρ_e (ρ_o) est la densité de sites vides (occupés) et $\langle s\rangle$ est la taille moyenne des amas détruits par les feux. Pourvu que ρ_e et ρ_o ne se comportent pas de manière singulière, on doit avoir

$$\langle s\rangle \asymp \frac{p}{f}. \tag{2}$$

Cette relation est parfaitement logique car le quotient p/f correspond au nombre d'arbres qui ont poussé entre deux incendies. Pour qu'une grande structure se forme, on doit donc avoir $p/f \gg 1$ et donc

$$1 \gg p \gg f. \tag{3}$$

Nous ne nous sommes pour l'instant occupés que des échelles de temps microscopiques, c'est-à-dire entre des étapes de l'algorithme (propagation, croissance, incendie). On doit de plus calculer l'échelle de temps macroscopique. La propagation des feux doit être instantanée en comparaison des échelles de temps de croissance (*i.e.* p^{-1}) et d'incendie (*i.e.* f^{-1}). Comme pour brûler une composante de taille N il faut environ N étapes, on doit donc avoir, d'après (2),

$$\frac{p}{f} \ll p^{-1}. \tag{4}$$

Finalement, en comparant (3) et (4), on doit avoir

$$\frac{p}{f} \ll p^{-1} \ll f^{-1}.$$

Cette dernière relation est connue sous le nom de *double séparation des échelles de temps* : le temps de propagation des incendies (*i.e.* p/f) est beaucoup plus petit que le temps caractéristique de croissance (*i.e.* $1/p$) qui lui-même est plus petit que le temps caractéristique d'apparition des incendies (*i.e.* $1/f$).

Le modèle peut se réécrire de la manière suivante : soient $L \in \mathbb{N}$ grand et $p, f > 0$ petits de sorte que $f/p \ll 1$ et $L \gg p/f$,

- sur chaque site, les arbres poussent avec taux 1 ;
- les allumettes tombent sur les sites occupés avec taux f/p et détruisent instantanément l'amas correspondant.

Les exposants critiques correspondants sont calculés par des arguments de champs moyens et leurs validations dans le modèle spatial sont vérifiées par simulations. Malgré cela, plusieurs travaux (par exemple [Gra93], [Hen93], [DCS94]) suggèrent des valeurs plus compliquées pour ces exposants et proposent des corrections sur les hypothèses posées dans [DS92]. Certains résultats dans le cas unidimensionnel obtenus non rigoureusement dans [DCS93] ont été prouvés plus tard par van den Berg et Jàrai [vdBJ05] et par Brouwer et Pennanen [BP06] dans un cadre un peu différent (voir la Section I.3 plus bas), tandis que les autres prédictions de [DCS93] ont été carrément infirmées.

Même si ce modèle est supposé exhiber des lois puissances (avec la distribution de la taille des forêts), il n'y a à priori aucune raison qu'il exhibe des propriétés d'invariance d'échelle, condition indispensable pour être classé dans les systèmes SOC. Des travaux plus récents, comportant des simulations plus poussées ([Gra02],[JP04]), jettent un doute sur le fait que les conditions énoncées plus haut conduisent vraiment à un comportement critique en deux dimensions.

I.3. Le processus des feux de forêts

On considère ici une généralisation du DS-MFF en temps continu.

Soit $G = (S, A)$ un graphe, où S est l'ensemble des sommets et A est l'ensemble des arêtes. Le graphe G n'est pas nécessairement fini. On note $E = \{0, 1\}^S$ l'espace des configurations. Pour $\eta \in E$, on dit que $\eta(i) = 0$ si le site $i \in S$ est *vide* et $\eta(i) = 1$ si le site i est *occupé par un arbre*. On dit que deux sites sont voisins s'il existe une arête entre eux. On appelle *forêt* une composante connexe de sites occupés. Pour $i \in S$ et $\eta \in E$, on définit $C(\eta, i)$ comme la forêt autour de i dans la configuration η (avec $C(\eta, i) = \emptyset$ si $\eta(i) = 0$). Soit $\lambda > 0$. Le $\lambda-$processus de feux de forêts ($\lambda-$PFF) est défini selon les règles suivantes : partant d'une configuration initiale quelconque,

- un arbre pousse sur chaque site vide avec taux 1 (une *graine* tombe et un arbre pousse *instantanément*) ;
- des *allumettes* (ou de la *foudre*) tombent sur chaque site occupé avec taux $\lambda > 0$ et brûlent *instantanément* la forêt correspondante.

Le cadre standard est celui où les processus qui gouvernent le système sont des processus de Poisson (les graines tombent selon un processus de Poisson de paramètre 1 tandis que les allumettes tombent selon un processus de Poisson de paramètre λ). Sauf mention explicite du contraire (parties I.3.3 et I.3.4), on se place dans ce cadre.

Une des difficultés dans l'étude du modèle de feux des forêts (et des systèmes critiques auto-organisés en général) est que l'interaction est non locale. Le processus, même s'il est markovien, n'est pas fellerien et certaines des techniques usuelles ne s'appliquent plus. Comme nous allons le voir (section I.3.1), en dimension 1, il n'y a pas de réels problèmes pour définir le $\lambda-$PFF : la taille des forêts reste toujours finie (il y a toujours des sites vides). La situation se complique en dimension supérieure : en l'absence de feux, les forêts deviennent infinies en temps fini (croissance sans feu = percolation par sites sur G). Mais les feux ont pour effet de détruire les amas *trop gros* : même s'il reste des forêts de taille arbitrairement grande, une allumette n'a qu'un effet local (dans le sens ou les allumettes qui tombent loin de l'origine n'affectent pas son état). Le manque de monotonie de ces modèles rend l'usage des techniques usuelles impossible : la monotonie permet de comparer deux processus qui partent de configurations différentes (par couplage). Ici, un processus dont la configuration initiale contient des arbres brûlera indubitablement plus tôt que le processus partant de la configuration initiale vide et l'ordre s'en trouvera inversé.

Le premier résultat mathématique a été établi par J. van den Berg et A. Járai [vdBJ05]. Ils calculent rigoureusement la densité asymptotique de sites vides lorsque $\lambda \to 0$ pour le processus des feux de forêts sur $\mathbb{Z}$. Ce résultat apparaissait pour la première fois dans le travail de [DCS93], mais les arguments étaient bancals (*ansatz* erroné). L'existence et l'unicité des processus de feux de forêts sur un graphe général n'ont été établies que très récemment par M. Dürre ([Dür06a], [Dür06b] et [Dür09]). L'existence d'une mesure invariante sur $\mathbb{Z}$ a été démontrée par Brouwer et Pennanen [BP06] puis étendue dans le cadre $\mathbb{Z}^d$ par A. Stahl [Sta12]. L'unicité n'a pu être démontrée que dans le cas où $\lambda = 1$ par X. Bressaud et N. Fournier [BF09]. Une question importante (pour rester dans l'esprit de SOC) est de comprendre le comportement du processus de feu de forêts lorsque $\lambda \to 0$, c'est-à-dire lorsqu'il y a de moins en moins d'allumettes qui tombent. En dimension 1, X. Bressaud et N. Fournier ont montré [BF10] que, après une renormalisation appropriée, le processus des feux de forêts converge vers un processus limite quand $\lambda \to 0$. Ils y décrivent la dynamique du processus limite (construction, existence et unicité) ainsi que la taille typique des forêts. Les auteurs ont étendu leurs résultats [BF13] dans le cas où les processus qui régissent la dynamique ne sont plus des processus de Poisson mais des processus de renouvellement stationnaires. L'étude du $\lambda-$PFF est intimement liée au graphe G sous-jacent. Plusieurs variantes du $\lambda-$PFF ont été étudiées. On en décrit quelques unes dans la partie I.3.4.

I.3.1. Existence et unicité du $\lambda-$processus de feux de forêts

Dans le cas où le graphe G est fini, l'existence et l'unicité des processus des feux de forêts est claire : on peut ordonner chronologiquement les temps auxquels tombent les graines et les allumettes et ainsi construire le processus *graphiquement*. Dans le cas du processus des feux de forêts sur $\mathbb{Z}$, un raisonnement simple arrive aux mêmes conclusions : si on veut construire le processus jusqu'à un temps T, en partant d'une configuration initiale avec une infinité de sites vides (on peut toujours le faire, car les forêts infinies sont immédiatement détruites par un feu), il suffit d'en trouver sur lesquels aucune graine ne tombe jusqu'à T. On peut alors partitionner $\mathbb{Z}$ en une collection (aléatoire) de sous-intervalles finis, qui n'interagissent pas entre eux (jusqu'au temps T). Le processus peut alors également se construire graphiquement.

Pour des graphes infinis plus généraux, cette approche ne fonctionne plus : l'existence et l'unicité du processus des feux de forêts sur G requiert des méthodes plus sophistiquées. En effet, il est naturel de considérer un *processus de percolation dynamique sur* G : considérons une famille $\{T_i : i \in S\}$ de variables aléatoires exponentielles indépendantes identiquement distribuées de paramètre 1. Posons $\eta_t(i) = 0$ si $t < T_i$, c'est-à-dire si aucune graine n'est tombée sur le site i au temps t, et $\eta_t(i) = 1$ si $t \geq T_i$: le processus $(\eta_t(i))_{t \geq 0, i \in S}$ est le processus de feux de forêts... sans feu ($\lambda = 0$). On l'appelle *processus de croissance*. Remarquons que, pour tout $t > 0$, l'ensemble $\{\eta_t(i) : i \in S\}$ est une percolation de paramètre $1 - e^{-t}$. Ainsi, un amas infini apparaît au temps critique t_c, défini par $1 - e^{-t_c} = p_c$.

Clairement, pour des petits temps, c'est-à-dire pour $t < t_c$, il n'y a que des composantes finies et le processus de feux de forêts peut être construit graphiquement. Dès que $t > t_c$, un amas infini peut (potentiellement) apparaître et la construction graphique du processus est impossible : l'état d'un site dans l'amas infini est directement influencé par une infinité d'autres sites. Bien sûr, les composantes géantes sont détruites par un feu et l'amas infini n'émerge en fait jamais. Tout ceci a été formalisé par M. Dürre ([Dür06a], Théorème 1) dans le cadre de graphes dont le degré des sommets est uniformément borné. Il montre l'existence du processus de feux de forêts pour tout $\lambda > 0$, pourvu que la configuration initiale ne contienne pas d'amas infini. Le même auteur s'est intéressé à l'unicité des processus des feux de forêts. Dans un premier temps [Dür06b], il montre le résultat pour un paramètre λ *assez grand* (dépendant évidemment du paramètre critique). Il généralise ce résultat dans ça thèse [Dür09] pour tout $\lambda > 0$ ([Dür09], Théorème 3) et toute configuration remplissant la condition de « cluster size bound » ([Dür09], Définition 7). La configuration initiale vide ou la percolation de paramètre $p < p_c$ remplissent par exemple cette condition. La question de l'unicité pour n'importe quelle configuration initiale est encore ouverte. De plus, en notant $G_n = (S_n, A_n)$, où S_n est l'ensemble des sommets qui sont à une distance plus petite que n de l'origine et A_n l'ensemble des arêtes associées, le processus des feux de forêts sur G_n converge presque sûrement vers le processus des feux de forêts sur G ([Dür09], Théorème 1).

I.3.2. Existence et unicité de mesures invariantes

L'existence d'une mesure stationnaire ne découle pas immédiatement des arguments de compacité usuels car le processus n'est pas Feller (à cause des intéractions à longue portée dûs à l'existence d'amas géants). Brouwer et Pennanen ([BP06], Proposition 5.1) montrent *à la main* l'existence d'au moins une mesure invariante stationnaire. De plus, ils définissent un *seuil maximal* s^λ_{max}, défini par $s^\lambda_{max} \log(s^\lambda_{max}) = 1/\lambda$, c'est-à-dire

$$s^\lambda_{max} \simeq \frac{1}{\lambda \log(1/\lambda)}$$

et montrent qu'il existe des constantes $0 < c < C$ tel que pour tout $\lambda \in (0\,,1)$, toute mesure μ_λ, stationnaire et invariante par translation pour le processus de feux de forêts sur $\mathbb{Z}$, et pour tout $x < (1/(\lambda \log(1/\lambda))^{1/3}$,

$$\frac{c}{(1+x)\log(1/\lambda)} \leq \mu_\lambda(|C(\eta,0)| = x) \leq \frac{C}{(1+x)\log(1/\lambda)}.$$

Récemment, en combinant les méthodes développées dans [Dür09] et [BP06], A. Stahl a étendu le résultat d'existence de mesures stationnaires et invariantes par translation dans le cas des processus des feux de forêts sur $\mathbb{Z}^d$ ([Sta12]Théorème 1).

X. Bressaud et N. Fournier ont démontré ([BF09], Théorème 1.1.) l'unicité de la mesure invariante dans le cas particulier $\lambda = 1$ (ils parlent de *processus d'avalanches*, les graines et les allumettes étant remplacées par des flocons de neige et des avalanches). Leur méthode se généralise aux modèles de feux de forêts de paramètre $\lambda > 1$ mais pas à ceux de paramètre $\lambda < 1$. L'unicité de la mesure invariante dans les cas $\lambda < 1$ reste encore à démontrer.

I.3.3. Asymptotiques des $\lambda-$processus de feux de forêts

Pour rester dans l'esprit du DS-modèle de feux de forêts, il est intéressant de regarder l'asymptotique des feux rares, c'est-à-dire de décrire le comportement du processus des feux de forêts quand $\lambda \to 0$. Lorsque $\lambda = 0$, le processus des feux de forêts est juste un processus de croissance. Ainsi, pour pouvoir voir l'effet d'un feu, il faut regarder le processus très longtemps. Pour espérer observer un comportement critique, il faut donc *accélérer* le temps. Peut-on trouver un processus limite dans des échelles correctes de temps et d'espace ? Quelle est la taille typique des forêts (elles tendent vers l'infinie, mais à quelle vitesse) ?

R. van den Berg et A. Járai ont étudié la densité de sites vides dans la limite $\lambda \to 0$. Ils montrent ([vdBJ05], Théorème 4) qu'il existe des constantes $0 < c_1 < C_1$ telles que pour toute configuration initiale, pour tout $\lambda > 0$ assez petit et tout t assez grand (de l'ordre d'au moins $\log(1/\lambda)$),

$$\frac{c_1}{\log(1/\lambda)} \leq \mathbb{P}\left[\eta_t^\lambda(0) = 0\right] \leq \frac{C_1}{\log(1/\lambda)}.$$

Il est amusant de remarquer que ce résultat avait été établi par Drossel et co-auteurs ([DCS93]), mais leur démonstration était basée sur des arguments non rigoureux. D'autres résultats conjecturés dans [DCS93] (sur la taille des amas) ont été infirmés dans [vdBJ05].

X. Bressaud et N. Fournier ont étudié [BF10] plus précisément le comportement asymptotique du processus du feux de forêts. Avant d'identifier un processus limite, il faut bien entendu décider du changement d'échelle à opérer. On suit ici leur raisonnement. Soit $(\eta_t^\lambda(i))_{t\geq 0, i\in\mathbb{Z}}$ le $\lambda-$PFF.

Définissons le temps caractéristique comme le temps au bout duquel environ une allumette tombe dans l'amas contenant 0. Notons $C(\eta_t^\lambda, 0)$ l'amas contenant 0 au temps t. Pour $\lambda > 0$ très petit et t pas trop grand, on peut négliger les feux et ainsi ne considérer que le processus de croissance. Comme les graines tombent selon un processus de Poisson de paramètre 1, chaque site est occupé avec probabilité $1 - e^{-t}$ et donc

$$\left|C(\eta_t^\lambda, 0)\right| \simeq e^t,$$

toujours en négligeant les feux *i.e.* pour t *assez petit*. Ainsi, comme chaque site brûle avec taux λ, la composante contenant 0 brûle avec taux $\lambda \left|C(\eta_t^\lambda, 0)\right| \simeq \lambda e^t$. On décide donc d'accélérer le temps par un facteur $\mathbf{a}_\lambda$ de sorte que $\lambda e^{\mathbf{a}_\lambda} = 1$, c'est-à-dire

$$\mathbf{a}_\lambda = \log(1/\lambda).$$

De cette manière, on a $\lambda \left|C(\eta_t^\lambda, 0)\right| \simeq 1$ et la probabilité qu'une allumette tombe dans la forêt contenant l'origine pendant l'intervalle de temps $[0\,, \mathbf{a}_\lambda]$ devrait tendre vers une valeur non triviale. Cependant, au bout d'un temps $\mathbf{a}_\lambda$, les composantes sont très grandes juste avant de brûler. Il convient alors de *contracter* l'espace, de sorte que environ une allumette tombe par unité d'espace et par unité de temps. Comme les allumettes tombent avec taux λ, on contracte l'espace d'un facteur

$$\mathbf{n}_\lambda = \lfloor 1/(\lambda \mathbf{a}_\lambda) \rfloor.$$

Cela veut dire que l'on identifie l'intervalle de temps $[0\,, \mathbf{a}_\lambda]$ à $[0\,, 1]$ et l'intervalle d'espace $[\![0\,, \mathbf{n}_\lambda]\!] \subset \mathbb{Z}$ à $[0\,, 1] \subset \mathbb{R}$. Les facteurs $\mathbf{a}_\lambda$ et $\mathbf{n}_\lambda$ définis ici apparaissaient déjà dans les travaux [vdBJ05] et [BP06].

Considérons à présent la nouvelle quantité

$$D_t^\lambda(0) = \frac{1}{\mathbf{n}_\lambda} C(\eta_{\mathbf{a}_\lambda t}^\lambda, 0),$$

qui n'est rien d'autre que l'amas qui contient 0 dans les nouvelles échelles de temps et d'espace. En reprenant les calculs menés plus haut, on a

$$D_t^\lambda(0) \simeq {\mathbf{n}_\lambda}^{-1} e^{\mathbf{a}_\lambda t} = \lambda^{1-t} \log(1/\lambda) \xrightarrow[\lambda \to 0]{} \begin{cases} 0 & \text{si } t < 1, \\ \infty & \text{si } t \geq 1. \end{cases}$$

Cela crée immédiatement une difficulté : quand $t \geq 1$, on espère que les feux agissent et rendent alors finie la taille des amas. Malheureusement, comme les feux ne peuvent que réduire la taille des amas, quand $t < 1$, la limite de $D_t^\lambda(0)$ est réellement 0 : à la limite, on a perdu des informations. Pour palier a ce manque, on introduit une nouvelle quantité censée décrire le comportement *microscopique* des amas, c'est-à-dire les amas qui ont une taille négligeable devant $\mathbf{n}_\lambda$.

Muni de ces deux grandeurs (taille des amas de l'ordre de $\mathbf{n}_\lambda$ et taille des amas beaucoup plus petits que $\mathbf{n}_\lambda$), ils montrent que le $\lambda-$processus des feux de forêts converge en loi lorsque $\lambda \to 0$ vers un processus limite. Ils décrivent précisément la dynamique de ce processus, montrent son unicité et qu'il peut être parfaitement simulé. De plus, en utilisant ce processus limite, ils montrent que pour t *assez grand* et λ *assez petit*, pour tout $0 < a < b < 1$,

$$\mathbb{P}\left[\left|C(\eta^{\lambda}_{\mathbf{a}_\lambda t}, 0)\right| \in [\lambda^{-a}, \lambda^{-b}]\right] \in [c(b-a), C(b-a)],$$

et pour tout $B > 0$

$$\mathbb{P}\left[\left|C(\eta^{\lambda}_{\mathbf{a}_\lambda t}, 0)\right| \geq \mathbf{n}_\lambda B\right] \in [ce^{-\kappa_2 B}, Ce^{-\kappa_1 B}],$$

pour certaines constantes $0 < c < C$ et $0 < \kappa_1 < \kappa_2$. Cela montre, de façon très faible, que pour tout $\lambda > 0$ assez petit et tout t assez grand (de l'ordre de $\log(1/\lambda)$), la taille des amas ressemble à

$$\mathbb{P}\left[\left|C(\eta^{\lambda}_{t}, 0)\right| = x\right] \simeq \frac{\alpha}{(x+1)\log(1/\lambda)} \mathbf{1}_{\{x \ll \mathbf{n}_\lambda\}} + \frac{\beta e^{-x/\mathbf{n}_\lambda}}{\mathbf{n}_\lambda},$$

avec $\alpha, \beta > 0$. Cela veut dire qu'il y a deux types d'amas : les amas microscopiques, décrits par une loi puissance, et des amas macroscopiques, décrits par une loi exponentielle. Il y a une *transition de phase* près de la *taille critique* $\mathbf{n}_\lambda = \lfloor 1/(\lambda \log(1/\lambda)) \rfloor$.

Il n'y a donc pas de comportement SOC : il y a bien une distribution en loi puissance mais elle ne décrit que les amas de taille très petite devant la taille critique.

Dans [BF13], les mêmes auteurs étendent leurs résultats aux processus de renouvellement stationnaires. Ils considèrent ainsi le cas où, en chaque site de $\mathbb{Z}$, le temps d'attente entre deux graines ne suit plus une loi exponentielle mais une loi ν_S et que le temps d'attente entre deux allumettes suit une loi ν_M. Pour étudier le processus des feux de forêts, défini de manière naturelle, des conditions sur les lois ν_S et ν_M sont imposées. Ils imposent à ν_S d'être soit à support borné, soit à variation lente, rapide ou régulière, c'est-à-dire de vérifier

$$\forall t > 0, \lim_{x \to \infty} \frac{\nu_S((x, \infty))}{\nu_S((tx, \infty))} = t^{\beta}, \tag{$H_S(\beta)$}$$

avec $\beta = \infty$ ou $\beta \in [0, \infty)$. Dans tous les cas, sous des conditions de renormalisation appropriées obtenues par des considérations heuristiques comme ci-dessus, ils montrent la convergence du processus des feux de forêts vers un processus limite qui est unique et qu'on peut construire graphiquement. Ils montrent qu'il y a quatre classes universelles selon que

- la loi ν_S est à support borné ;
- la queue de distribution de ν_S décroît rapidement ;
- la queue de distribution de ν_S est polynomiale ;
- la queue de distribution de ν_S est logarithmique.

Ils décrivent, dans chaque cas, la taille typique des amas. Dans [BF13] comme dans [BF10], on n'observe pas de criticalité. Il est remarquable que leurs résultats ne dépendent que d'une hypothèse assez faible sur la loi des temps d'attente. En effet, en observant que $x \mapsto \nu_S((x\,,\infty))$ est décroissante, lipschitzienne et convexe, l'hypothèse $(H_S(\beta))$ est automatiquement vérifiée par la plupart des lois.

Il n'y a pas encore de résultats précis sur l'asymptotique des feux rares sur des graphes plus généraux. Comme pour montrer l'existence du $\lambda-$PFF, Section I.3.1, les démonstrations se compliquent dès que la dimension du réseau augmente et font appel à des arguments très fins de percolation (géométrie des composantes infinies). Un premier résultat sur le réseau $\mathbb{Z}^2$ a été obtenu par J. van den Berg et R. Brouwer [vdBB06]. Définissons t_c par la relation $1-e^{-t_c} = 1/2 = p_c(2)$. Ils montrent que, conditionnellement à une conjecture démontrée depuis par Kiss, Manolescu et Sidoravicius [KMS13], il existe $t > t_c$ tel que pour tout $m \geq 1$,

$$\liminf_{\lambda\to 0}\liminf_{n\to\infty}\mathbb{P}\left[\begin{matrix}\text{un arbre de } [\![-m,m]\!]^2 \text{ brûle avant } t\\ \text{dans le processus de feux de forêts défini sur } S_n{=}[\![-n,n]\!]^2\end{matrix}\right] \leq \frac{1}{2}.$$

Cette dernière inégalité est plutôt surprenante : intuitivement, on peut espérer que pour $t > t_c$ fixé – le processus de croissance sans feu est alors la percolation sur $\mathbb{Z}^2$ avec probabilité $1-e^{-t} > p_c(2)$, il y a donc un unique amas infini – si on fait *simultanément* tendre λ vers 0 et m vers l'infini, la probabilité qu'un arbre à une distance plus petite que m de l'origine brûle avant t tende vers 1.

L'étude de *processus des feux de forêts modifiés* peut donner des réponses ou, tout du moins, des indications sur les comportements du *vrai* processus des feux de forêts. On décrit dans la prochaine Section quelques processus de feux de forêts modifiés.

I.3.4. Quelques modèles en relation avec le $\lambda-$processus de feux de forêts

On a vu que l'étude des processus des feux de forêts est rendue difficile à cause des interactions à longue portée et du manque de monotonie. Pour contourner ce problème, on peut étudier d'autres modèles qui contournent ces problèmes.

I.3.4.1. Percolation auto-destructrice

Introduite par J. van den Berg et R. Brouwer dans [vdBB04], la percolation auto-destructrice (*self-destructive percolation*) est définie de la manière suivante. Fixons nous un graphe infini $G = (S, A)$, où S est l'ensemble des sommets et A est l'ensemble des arêtes. Pour $\delta \geq 0$, considérons la percolation par sites de paramètre p (on ouvre chaque site avec probabilité p, indépendamment les uns des autres). Fermons tous les sites se trouvant dans les (potentielles) composantes connexes infinies : on dit que les amas infinis sont brûlés. Finalement, ouvrons tous les sites fermés avec probabilité δ, indépendamment de tous les choix précédents. On appelle $\mathbb{P}_{p,\delta}$ la mesure gouvernant la configuration ainsi obtenue et $\theta(p,\delta)$ la $\mathbb{P}_{p,\delta}-$probabilité qu'un site donné (appelé origine) se trouve dans un amas infini.

On définit

$$\delta_c(p) \coloneqq \inf\{\delta : \theta(p,\delta) > 0\}$$

et on pose $p_c = p_c(G)$, le point critique pour la percolation par sites. Il est facile de voir que $\theta(p,\delta)$ est nul si $p < p_c$ tandis que $\theta(p,\delta)$ est positif si et seulement si la configuration finale contient presque sûrement au moins un amas infini (c'est-à-dire $p > p_c$). La question intéressante est donc de connaître le comportement de $\delta_c(p)$ quand $p \downarrow p_c$. Dans leur publication originale [vdBB04], VAN DEN BERG et BROUWER ont conjecturé que, pour un graphe plan, δ_c est borné uniformément loin de 0 quand $p > p_c$, c'est-à-dire qu'il existe existe $\delta_0 > 0$ tel que pour tout $p > p_c$,

$$\theta(p, \delta_0) = 0. \tag{I.3.1}$$

La conjecture est plutôt surprenante : quand p est vraiment proche de p_c, l'amas infini est vraiment fin et après l'avoir brûlé, on peut espérer qu'il ne faille ouvrir que quelques sites pour obtenir à nouveau un amas infini.

Il se trouve que la réponse dépend crucialement de la géométrie du graphe. Elle a été infirmée pour $\mathbb{Z}^d$ avec d assez grand dans [ADCKS13]. En dimension 2, il a été montré ([vdBB04], Proposition 3.1) que $\delta_c(p) > 0$ pour $p > p_c$. Ce résultat a été renforcé par VAN DEN BERG et DE LIMA [vdBdL09] qui ont montré que $\delta_c(p) \geq (p - p_c)/p$. Récemment KISS, MANOLESCU et SIDORAVICIUS [KMS13] ont démontré cette conjecture dans le cas du réseau $\mathbb{Z}^2$.

I.3.4.2. Processus de feux de forêts en champ moyen

Nous présentons ici un modèle de feux de forêts en champ moyen. Le point de vue adopté est un peu différent. Il a été étudié par B. RÀTH et B. TÒTH dans [RT09] et est intimement lié au graphe aléatoire d'ERDÖS-RÉNYI. Un comportement critique auto-organisé a été rigoureusement établi.

Commençons par rappeler quelques résultats bien connus sur le graphe aléatoire d'ERDÖS-RÉNYI, qui peut être vu comme une percolation sur le graphe complet a n sommets. On note $G_n = (S_n, A_n)$ le graphe complet à n sommets ($|S_n| = n$, tous les sommets sont joints par une arête). On considère le graphe de façon dynamique : au temps $t = 0$, il y a n sommets et aucune arête. Les arêtes s'ouvrent, indépendamment, avec taux $1/n$. On définit la concentration des amas de masse $k \geq 1$ au temps $t \geq 0$

$$v_k^n(t) = \frac{\text{nombre de composantes de taille } k \text{ au temps } t}{n}.$$

À la limite $n \to \infty$, il y a une transition de phase : une composante géante contenant une fraction positive de tous les sommets émerge au temps critique $t_c = 1$. Une façon de formaliser tout cela est de dire que $v_k^n(t)$ converge en probabilité vers une limite déterministe $v_k(t)$, quand $n \to \infty$, où la limite satisfait

$$\sum_{k \geq 1} v_k(t) \begin{cases} = 1 & \text{si } t \leq 1, \\ < 1 & \text{si } t > 1. \end{cases}$$

Le défaut de masse pour $t > 1$ est du à l'apparition d'une composante géante, de taille de l'ordre de n. De plus, pour $t < 1$, $v_k(t)$ décroît exponentiellement vite avec k tandis que, au temps critique $t_c = 1$, on a

$$v_k(t_c) \sim ck^{-3/2}.$$

Le modèle est ainsi sous-critique si $t < 1$, critique pour $t = 1$ et sur-critique pour $t > 1$.

On modifie à présent le mécanisme de sorte que la composante géante n'apparaisse jamais : soit $\lambda(n)$ une fonction telle que $1/n \ll \lambda(n) \ll 1$. Supposons que des allumettes tombent sur chaque sommet, indépendamment, avec taux $\lambda(n)$. Quand une allumette tombe sur un sommet, la composante le contenant est cassée en sommets individuels, c'est-à-dire que toutes les arêtes sont fermés. Heuristiquement, ce mécanisme devrait interdire les composantes de taille de l'ordre de n ($n\lambda(n) \gg 1$). Inversement, la relation $\lambda(n) \ll 1$ montre que les amas de petites tailles ne sont pas touchés par les feux et peuvent donc grandir plus au moins comme dans le modèle d'Erdös-Rényi. L'heuristique suggère qu'après le temps critique $t_c = 1$, le système reste critique pour toujours. Ràth et Tòth montrent [RT09] que c'est effectivement le cas : en notant $\bar{v}_k^n(t)$ la proportion (concentration) d'amas de taille k au temps t,

- la suite $(\bar{v}_k^n(t))_{n\in\mathbb{N}}$ converge en probabilité vers une fonction déterministe

$$\bar{v}_k(t) := \lim_{n\to\infty} \bar{v}_k^n(t);$$

- si $t \leq t_c$, $\bar{v}_k(t) = v_k(t)$, où $(v_k(t))_{k\geq 1}$ est définie plus haut ;

- si $t \geq t_c$, on a $\sum_{l\geq k} \bar{v}_l(t) \asymp k^{-1/2}$.

Le modèle exhibe un comportement SOC dans le sens où avant t_c, il n'y a pas de composante géante tandis qu'après t_c, la distribution des tailles est dans un sens critique pour toujours.

I.4. Travaux de thèse

On étudie un *processus de feux de forêts avec propagation non instantanée.* Posons $E = \{0, 1, 2\}^{\mathbb{Z}}$. Soit $\eta \in E$, on dit que $\eta(i) = 0$ si le site $i \in \mathbb{Z}$ est *vide*, $\eta(i) = 1$ si le site i est *occupé par un arbre* et $\eta(i) = 2$ si le site i est *en feu.* On appelle *forêt* une composante connexe de sites occupés. Pour $i \in \mathbb{Z}$ et $\eta \in E$, on définit $C(\eta, i)$ comme la forêt autour de i dans la configuration η (avec $C(\eta, i) = \emptyset$ si $\eta(i) = 0$ ou si $\eta(i) = 2$). Soient $\lambda \in (0\,, 1)$ et $\pi \geq 1$. On définit le $(\lambda, \pi)-$processus de feux de forêts ($(\lambda, \pi)-$PFF) de la manière suivante : sur $\mathbb{Z}$, partant d'une configuration initiale vide,

- sur chaque site, des graines tombent selon un processus de Poisson de paramètre 1. Si le site est vide, un arbre pousse instantanément ;

- sur chaque site, des allumettes tombent selon un processus de Poisson de paramètre λ. Si le site est occupé par un arbre, l'arbre brûle...

- ... pendant un temps exponentiel de paramètre π, avant de se propager à ses deux voisins. S'ils sont occupés, ils brûlent. L'arbre devient alors cendre et le site redevient vide.

On note $(\eta_t^{\lambda,\pi}(i))_{t\geq 0, i\in\mathbb{Z}}$ le processus ainsi défini. Comme dans la Section I.3.1, on peut facilement montrer qu'un tel processus existe (sur $\mathbb{Z}$).

D'un point de vue critique (voir section I.2), le cas intéressant est celui où

$$\lambda \ll 1 \ll \pi,$$

c'est-à-dire lorsque la fréquence d'apparition des allumettes tend vers 0 et que la vitesse de propagation des feux tend vers l'infini.

Comme décrit dans la section I.3.3, en observant le processus dans un intervalle de temps fini $[0\,, T]$, aucun comportement critique n'émergera (car $\lambda \to 0$). Pour pouvoir observer un comportement critique, il faut changer d'échelle de temps. En remarquant que le calcul heuristique effectué en section I.3.3 ne fait intervenir que le processus de croissance (on néglige les feux), un raisonnement analogue implique donc que l'échelle de temps doit être de l'ordre de $\mathbf{a}_\lambda = \log(1/\lambda)$ tandis que l'échelle d'espace doit être de l'ordre de $\mathbf{n}_\lambda = \lfloor 1/(\lambda \log(1/\lambda))\rfloor$. Évidemment, comme l'heuristique est faite en négligeant les feux (et donc la propagation des feux), les échelles ne dépendent pas du paramètre π.

On définit alors l'amas contenant 0 dans nos nouvelles échelles,

$$D_t^{\lambda,\pi}(0) = \frac{1}{\mathbf{n}_\lambda} C(\eta_{\mathbf{a}_\lambda t}^{\lambda,\pi}, 0).$$

La difficulté aperçue en section I.3.3 est encore présente : en l'absence de feu, on a

$$D_t^{\lambda,\pi}(0) \simeq {\mathbf{n}_\lambda}^{-1} e^{\mathbf{a}_\lambda t} = \lambda^{1-t}\log(1/\lambda) \xrightarrow[\lambda\to 0]{} \begin{cases} 0 & \text{si } t < 1, \\ \infty & \text{si } t \geq 1. \end{cases}$$

Pour $t \geq 1$, on espère que les feux agissent et rendent alors finie la taille des amas. Malheureusement, la limite de $D_t^{\lambda,\pi}(0)$ est réellement 0 quand $t < 1$, car les feux ne peuvent que réduire la taille des forêts. Pour palier à ce défaut, on définit,

$$\mathbf{m}_\lambda = \left\lfloor \frac{1}{\lambda \mathbf{a}_\lambda^2} \right\rfloor$$

et on introduit, pour $t \geq 0$,

$$K_t^{\lambda,\pi}(0) = \frac{\left|\left\{ i \in [\![-\mathbf{m}_\lambda\,, \mathbf{m}_\lambda]\!] : \eta_{\mathbf{a}_\lambda t}^{\lambda,\pi}(i) = 0 \right\}\right|}{2\mathbf{m}_\lambda + 1} \in [0\,, 1],$$

$$Z_t^{\lambda,\pi}(0) = \frac{-\log(K_t^{\lambda,\pi}(x))}{\log(1/\lambda)} \wedge 1 \in [0\,, 1].$$

Observons que $\mathbf{m}_\lambda \ll \mathbf{n}_\lambda$ mais que pour $t < 1$, en se rappelant les calculs effectués plus haut, en négligeant les feux, on a

$$\left|C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}, 0)\right| \simeq e^{\mathbf{a}_\lambda t} = \lambda^{-t} \ll \mathbf{m}_\lambda.$$

Ainsi, $K^{\lambda,\pi}_t(0)$ peut être interprété comme la *densité locale de sites vides autour de* 0 (locale car $\mathbf{m}_\lambda \ll \mathbf{n}_\lambda$). De plus, on espère que pour $t < 1$, on ait $K^{\lambda,\pi}_t(0) \simeq \lambda^t$ d'où $Z^{\lambda,\pi}_t(0) \simeq t$.

On décrit alors le comportement du $(\lambda, \pi)-$PFF autour de l'origine à travers le processus $(D^{\lambda,\pi}_t(0), Z^{\lambda,\pi}_t(0))_{t\geq 0, x\in\mathbb{R}}$. L'idée principale est que pour $\lambda > 0$ très petit,

- si $Z^{\lambda,\pi}_t(0) = z \in (0\,, 1)$, alors $\left|D^{\lambda,\pi}_t(0)\right| \simeq 0$ et l'amas contenant 0 est *microscopique*, dans le sens où la taille de l'amas avant changement d'échelle est très petite comparée à $\mathbf{n}_\lambda$;

- si $Z^{\lambda,\pi}_t(0) = 1$, alors $\left|D^{\lambda,\pi}_t(0)\right| = [a\,, b]$: l'amas qui contient l'origine est *macroscopique* et la taille de l'amas avant changement d'échelle est de l'ordre $\mathbf{n}_\lambda\, |b - a|$.

On cherche à présent à décrire le comportement d'un feu. Imaginons qu'une zone $[\![\lfloor a\mathbf{n}_\lambda\rfloor\,, \lfloor b\mathbf{n}_\lambda\rfloor]\!]$, avec $a < 0 < b$, soit complètement remplie à un certain temps $\mathbf{a}_\lambda t_0$ et qu'une allumette tombe en 0 au temps $\mathbf{a}_\lambda t_0$. Comme le feu met un temps de l'ordre de $1/\pi$ à se propager à son voisin, en négligeant tous les autres phénomènes, il atteindra le site $\lfloor b\mathbf{n}_\lambda\rfloor$ au temps

$$\mathbf{a}_\lambda t_0 + \frac{\lfloor b\mathbf{n}_\lambda\rfloor}{\pi}.$$

Si $\mathbf{n}_\lambda/\pi \gg \mathbf{a}_\lambda$ alors, dans les échelles de temps considérées, le feu ne pourra pas atteindre le site $\lfloor b\mathbf{n}_\lambda\rfloor$ tandis que si $\mathbf{n}_\lambda/\pi \ll \mathbf{a}_\lambda$, le feu atteindra le point $\lfloor b\mathbf{n}_\lambda\rfloor$ *très rapidement*. Si $\mathbf{n}_\lambda/\pi \simeq p\mathbf{a}_\lambda$, avec $p > 0$, le feu atteindra le site $\lfloor b\mathbf{n}_\lambda\rfloor$ en un temps de l'ordre de $bp\mathbf{a}_\lambda$ (le temps caractéristique).

L'objectif est donc d'étudier la convergence du $(\lambda, \pi)-$PFF lorsque λ tend vers 0 et π vers l'infini dans les différents régimes, c'est-à-dire lorsque $\lambda \to 0$ et $\pi \to \infty$ avec

$$\frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} \sim \frac{1}{\lambda \mathbf{a}_\lambda^2 \pi} \to p \in [0\,, \infty) \cup \{\infty\}.$$

On dit que la convergence a lieu dans le régime

- rapide si $p = 0$;

- intermédiaire si $p \in (0\,, \infty)$;

- lent si $p = \infty$.

Décrivons à présent les caractéristiques principales des différents régimes.

Étude de la convergence du $(\lambda, \pi)-$**PFF dans le régime lent**

Dans cette partie, on s'intéresse à la convergence du $(\lambda, \pi)-$PFF dans le régime où $\lambda \to 0$ et $\pi \to \infty$ avec

$$\frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} \to \infty.$$

Ce régime est un peu particulier car il n'y a pas, asymptotiquement, d'interaction entre les sites : si une allumette tombe sur un site $\lfloor \mathbf{n}_\lambda x_0 \rfloor$ au temps $\mathbf{a}_\lambda t_0$, le feu n'affectera les sites que *localement* dans le sens où pour tout $\varepsilon > 0$ et tout $|x - x_0| > \varepsilon$, pour λ petit et π grand tels que $\mathbf{n}_\lambda/(\mathbf{a}_\lambda \pi)$ soit grand, le feu n'atteindra pas le site $\lfloor \mathbf{n}_\lambda x \rfloor$ dans l'intervalle de temps $[0\,, \mathbf{a}_\lambda T]$.

Il reste cependant une petite subtilité : on sait que le feu n'affecte pas les sites se trouvant à une distance de l'ordre de $\mathbf{n}_\lambda$. Que se passe-t-il pour les sites *proches* ? Lorsque $\lambda \to 0$ et $\pi \to \infty$ dans le régime lent, on suppose l'existence et on définit

$$z_0 := \sup\{s \geq 0 : 1/(\lambda^s \mathbf{a}_\lambda \pi) \to 0\} \in [0\,, 1].$$

Rappelons que pour $t < 1$, la taille des forêts est de l'ordre de λ^{-t} et qu'un feu démarrant en $i_0 \in \mathbb{Z}$ au temps $\mathbf{a}_\lambda t_0$ atteint le site $i \in \mathbb{Z}$ au temps $\mathbf{a}_\lambda t_0 + |i - i_0|/\pi$. Le paramètre z_0 est donc défini de sorte que si une allumette tombe dans une zone A alors

- si $|A| \ll \lambda^{-z_0}$, le feu se propage très rapidement (*instantanément*) dans la composante A et s'éteint ;
- si $|A| \gg \lambda^{-z_0}$, la forêt est trop grosse pour être brûlée entièrement dans nos échelles de temps : le feu brûle pour toujours.

Pour $z_0 \in [0\,, 1]$, on étudiera la convergence du $(\lambda, \pi)-$PFF dans le régime $\mathcal{R}(\infty, z_0)$ c'est-à-dire dans le régime où $\lambda \to 0$ et $\pi \to \infty$ avec

$$\frac{1}{\lambda \mathbf{a}_\lambda \pi} \to \infty \text{ et } \frac{\log(\pi)}{\log(1/\lambda)} \to z_0.$$

Finalement, partant d'une configuration initiale vide, pour $t \in [0\,, 1)$, si aucune allumette ne tombe, la taille des amas est de l'ordre de $e^{\mathbf{a}_\lambda t} = \lambda^{-t}$. Ainsi, si une allumette tombe à l'instant $\mathbf{a}_\lambda t_0$ avec $t_0 < z_0$, le feu se propage dans une zone de taille $\lambda^{-t_0} \ll \lambda^{-z_0}$ pendant un temps d'environ

$$1/(\lambda^{-t_0}\pi) \ll \mathbf{a}_\lambda.$$

Comme le $(\lambda, \pi)-$PFF est un processus de Markov, le temps que met la zone à se remplir à nouveau est (intuitivement) de l'ordre de $\mathbf{a}_\lambda t_0$. Si maintenant une allumette tombe à l'instant $\mathbf{a}_\lambda t_0$ avec $t_0 > z_0$, l'allumette tombe dans une zone A de taille

$$|A| \simeq \lambda^{-t_0} \wedge \mathbf{n}_\lambda \gg \lambda^{-z_0}.$$

Le feu n'atteint jamais le bord de la zone. Comme il n'est pas affecté par d'autres feux, il brûle pour toujours.

Ainsi, dans nos nouvelles échelles, le processus limite doit se comporter de la sorte :

- pour presque tous les sites, les arbres poussent sans être affectés par des feux. Au temps $t \in [0\,,1)$, toutes les zones sont microscopiques et sont décrites par le processus $(Z_t^{\lambda,\pi}(x))_{t\geq 0, x\in\mathbb{R}}$. Au temps $t=1$, les zones macroscopiques émergent ;
- localement (à l'endroit où tombent des allumettes), des feux démarrent. Si la zone n'est pas trop grosse, c'est-à-dire si l'allumette tombe à l'instant $t \in [0\,, z_0)$, la forêt n'a pas eu le temps de trop grandir et est détruite instantanément. Le feu s'éteint et crée une barrière de hauteur t (le temps que la zone vidée se remplisse à nouveau). Si l'allumette tombe après z_0, comme la forêt n'a pas été affectée par des feux (la probabilité que deux allumettes tombent très proche est très petite), le feu continue de brûler pour toujours (dans nos échelles de temps).

Pour tout $z_0 \in [0\,,1]$, on définit un processus limite et on montre la convergence du $(\lambda,\pi)-$PFF vers ce processus limite lorsque $\lambda \to 0$ et $\pi \to \infty$ dans le régime $\mathcal{R}(\infty, z_0)$.

Étude de la convergence du $(\lambda,\pi)-$PFF dans le régime rapide

Intéressons-nous à présent à la convergence du $(\lambda,\pi)-$PFF dans le régime rapide, c'est-à-dire lorsque $\lambda \to 0$ et $\pi \to \infty$ avec $\mathbf{n}_\lambda/(\mathbf{a}_\lambda\pi) \to 0$. On sait que si une allumette tombe :

- dans une zone A de taille $|A| \ll \mathbf{n}_\lambda$, alors le feu mettra un temps
$$\frac{|A|}{\pi} \ll \frac{\mathbf{n}_\lambda}{\pi} \ll \mathbf{a}_\lambda$$
pour traverser la zone A : à la limite, dans nos nouvelles échelles, le feu se propagera *instantanément*.
- dans une zone $A = [\![\lfloor a\mathbf{n}_\lambda\rfloor\,, \lfloor b\mathbf{n}_\lambda\rfloor]\!]$, avec $a < b$, alors le feu mettra un temps (au plus)
$$\frac{(b-a)\mathbf{n}_\lambda}{\pi} \ll \mathbf{a}_\lambda$$
pour traverser la zone A : à la limite, dans nos nouvelles échelles, le feu se propagera aussi *instantanément*.

Ainsi, à la limite, dans le régime $\mathcal{R}(0)$, tout se passe comme si le feu se propageait instantanément : dans le processus discret, quand une allumette tombe dans une zone, le temps que le feu met à se propager est négligeable devant $\mathbf{a}_\lambda$, c'est-à-dire qu'après changement d'échelle, le feu se propage instantanément. En comparant la dynamique de ce processus (pour λ petit et π grand de sorte que $\mathbf{n}_\lambda/(\mathbf{a}_\lambda\pi)$ soit assez proche de 0) avec le $\lambda-$processus de feux de forêts, défini dans la section I.3.3 (processus avec propagation instantanée *i.e.* « $\pi = \infty$ »), on espère que le $(\lambda,\pi)-$FFP converge vers le même processus limite défini dans [BF10] : les différences dues à la propagation du feu dans le $(\lambda,\pi)-$FPP ne se répercutent pas à la limite. Dans ce régime, l'interaction est à longue portée.

On montre que le $(\lambda,\pi)-$PFF converge effectivement en loi vers le processus limite défini dans [BF10], lorsque $\lambda \to 0$ et $\pi \to \infty$ dans le régime $\mathcal{R}(0)$.

Étude de la convergence du $(\lambda, \pi)-$PFF dans le régime intermédiaire

Soit $p \in (0\,,\infty)$. On s'intéresse enfin à la convergence du $(\lambda, \pi)-$PFF quand $\lambda \to 0$ et $\pi \to \infty$ avec

$$\frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} \to p.$$

Comme dans les parties précédentes, étudions l'effet des feux.

- Si une allumette tombe dans une zone A de taille $|A| \ll \mathbf{n}_\lambda$, alors le feu mettra un temps
$$\frac{|A|}{\pi} \ll \mathbf{a}_\lambda$$
pour traverser la zone A : à la limite, dans nos nouvelles échelles, comme dans le cas du régime rapide, le feu se propage *instantanément*. Ici, l'effet des feux microscopiques, c'est-à-dire des feux qui se déclarent dans une zone microscopique, est le même que dans le régime rapide.

- Le cas où une allumette tombe dans une zone $A = [\![\lfloor a\mathbf{n}_\lambda \rfloor\,, \lfloor b\mathbf{n}_\lambda \rfloor]\!]$, avec $a < b$, est un peu différent : si l'allumette tombe par exemple en $\lfloor \mathbf{n}_\lambda x_0 \rfloor$, avec $x_0 \in (a\,, b)$, alors le feu mettra un temps
$$\frac{(b - x_0)\mathbf{n}_\lambda}{\pi} \simeq p(b - x_0)\mathbf{a}_\lambda$$
à rejoindre le bord de la zone. À la limite, dans nos nouvelles échelles, le feu met un temps p à traverser une zone de taille 1.

En combinant le comportement des feux microscopiques et des feux macroscopiques, on peut alors facilement distinguer un processus limite. On définit ce processus et on montre la convergence du $(\lambda, \pi)-$PFF vers ce processus limite dans le Chapitre 5.

I.5. Perspectives

Au chapitre 3 de cette thèse, nous présentons des travaux en cours. Il s'agit d'une étude du *processus de feux de forêts en environnement aléatoire*. Les démonstrations des théorèmes n'ont pour l'instant pas été écrites. Le Chapitre 3 n'est constitué que de preuves heuristiques, nous espérons qu'elles soient tout de même convaincantes.

On utilise les notations classiques des processus de feux de forêts (définies à la Section I.3.3). Soit ν une probabilité portée par $\mathbb{R}_+$ (c'est-à-dire telle que $\nu((-\infty\,,0]) = 0$) et $(\kappa_i)_{i\in\mathbb{Z}}$ une suite de variables aléatoires indépendantes et identiquement distribuées selon ν. Soit $\lambda \geq 0$. On définit le $(\lambda, \nu)-$processus de feux de forêts en environnement aléatoire ($(\lambda, \nu)-$PFFEA) de la manière suivante : partant d'une configuration initiale vide,

- sur chaque site $i \in \mathbb{Z}$, des graines tombent selon un processus de Poisson de paramètre κ_i. Si le site est vide, un arbre pousse instantanément ;

- sur chaque site, des allumettes tombent selon un processus de Poisson de paramètre λ. Si le site est occupé, le feu détruit instantanément la composante connexe correspondante de sites occupés.

On note $(\eta_t^\lambda(i))_{t\geq 0, i\in\mathbb{Z}}$ le processus ainsi défini. Comme dans la Section I.3.1, on montre facilement l'existence et l'unicité d'un tel processus (sur $\mathbb{Z}$).

Pour $t \geq 0$, on définit la transformée Laplace de la loi ν,

$$G(t) = \int_{\mathbb{R}_+} e^{-xt} \nu(\mathrm{d}x).$$

Clairement, $G(0) = 1$, G est strictement décroissante, convexe, analytique sur $(0\,,\infty)$ et, comme $\nu(0) = 0$, $G(t) \xrightarrow[t\to\infty]{} 0$. Dans cette partie, on suppose de plus que $1/G$ est soit à variation lente, soit à variation rapide, soit à variation régulière d'indice $\beta > 0$, c'est-à-dire que

$$\forall t > 0,\ \lim_{x\to\infty} \frac{G(x)}{G(xt)} = t^\beta,$$

où par convention on pose

$$t^\infty = \begin{cases} 0 & \text{si } t \in (0\,,1), \\ 1 & \text{si } t = 1, \\ \infty & \text{si } t > 1. \end{cases}$$

Remarquons que cette hypothèse n'est pas vraiment restrictive à la vue des propriétés de G : la plupart des lois la satisfont.

Le cas intéressant reste bien entendu l'asymptotique des feux rares *i.e.* l'étude de la limite $\lambda \to 0$. Pour définir une échelle de temps appropriée, le raisonnement est un peu différent de celui de la partie précédente. Comme les graines tombent sur le site $i \in \mathbb{Z}$ selon un processus de Poisson de paramètre κ_i, en négligeant les feux, le site i sera occupé à l'instant t avec probabilité $\mathbb{E}\left[1 - e^{-\kappa_i t}\right] = 1 - G(t)$. Un calcul grossier montre que pour tout $t > 0$,

$$\mathbb{E}\left[\left|C(\eta_t^\lambda, 0)\right|\right] \simeq 1/G(t).$$

Comme chaque site brûle avec taux $\lambda > 0$, on décide d'accélérer le temps d'un facteur $\mathbf{a}_\lambda$ tel que

$$\lambda \int_0^{\mathbf{a}_\lambda} \frac{1}{G(s)}\,\mathrm{d}s = 1,$$

de sorte que la probabilité qu'une allumette tombe dans l'amas contenant 0 pendant l'intervalle de temps $[0\,,\mathbf{a}_\lambda]$ tende vers une valeur non triviale. On montre facilement que

$$\mathbf{a}_\lambda \xrightarrow[\lambda\to 0]{} \infty \text{ et } \lambda\mathbf{a}_\lambda \xrightarrow[\lambda\to 0]{} 0.$$

Comme les composantes sont très grandes juste avant de brûler, on doit contracter l'espace. On définit $\mathbf{n}_\lambda$, comme d'habitude, par la relation

$$\mathbf{n}_\lambda = \lfloor 1/(\lambda\mathbf{a}_\lambda) \rfloor,$$

de sorte que, après changement d'échelle, environ une allumette tombe par unité de temps et d'espace.

On distinguera alors trois cas :

- dans un premier temps, on étudiera le $(\lambda, \nu)-$PFFEA pour les lois ν dont l'inverse de la transformée de Laplace est à variation rapide ;
- dans un deuxième temps, on étudiera le $(\lambda, \nu)-$PFFEA pour les lois ν dont l'inverse de la transformée de Laplace est à variation régulière d'indice $\beta > 0$;
- finalement, on étudiera le $(\lambda, \nu)-$PFFEA pour les lois ν dont l'inverse de la transformée de Laplace est à variation lente.

Ce qui est remarquable au premier abord est l'universalité des modèles limites. On n'impose qu'une condition – assez faible – sur le comportement en l'infini de la transformée de Laplace de la loi ν. En fait, les théorèmes Taubérien font le lien entre le comportement en l'infini de la transformée de Laplace de ν et le comportement en 0 de ν. On étudiera notamment les exemples où

- $\inf(\text{supp }(\nu)) = 0$ et $1/G$ est à variation régulière d'indice $\beta \in (0\,,\infty)$;
- $\inf(\text{supp }(\nu)) = 0$ et $1/G$ est à variation rapide ;
- $\inf(\text{supp }(\nu)) = x_0 > 0$: dans ce cas, $1/G$ est forcément à variation rapide.

Les démonstrations des théorèmes sont longues et assez fastidieuses. Elles ne sont pour l'instant pas écrites. On tâchera de convaincre le lecteur en donnant des preuves heuristiques. On montrera notamment que le processus limite trouvé dans [BF10], dans [BF13] cas $\beta = \infty$, dans la Section I.4 cas propagation rapide et celui espéré dans le cas où $1/G$ est à variation rapide dans la présente partie est le même. On tâchera d'en expliquer la raison.

I.6. Conclusion

On a présenté ici des raffinements du processus des feux de forêts sur $\mathbb{Z}$ défini dans [BF10] (cas poissonnien, avec propagation instantanée en milieu déterministe). Le modèle limite trouvé dans ce travail est universel : il correspond aussi au cas du processus limite dans

- [BF13], cas des processus de renouvellement avec délais à décroissance rapide ;
- [LC15], cas du régime rapide ;
- le cas $\beta = \infty$ dans le modèle en environnement aléatoire, décrit dans la Section I.5, avec pas (ou peu) de sites *arbitrairement lents*.

Néanmoins, que ce soit dans le cas des processus de renouvellement (avec délais à décroissance lente ou polynomiale), dans le cas des processus avec propagation non instantanée (cas des régimes intermédiaire et lent) ou dans le cas des processus en environnement aléatoire (quand il y a suffisamment de sites *lents i.e.* avec très peu de graines), l'étude du processus des feux de forêts fait apparaître d'autres limites.

II. Asymptotics of the one dimensional forest-fire processes with non-instaneous propagation

Sommaire

Résumé

On considère le modèle suivant de feux de forêts sur $\mathbb{Z}$, où chaque site a trois états possibles : *vide*, *occupé* ou *en feu*. Un site vide devient occupé avec taux 1. Sur chaque site, des allumettes tombent avec taux λ. Si le site est occupé, il brûle pendant un temps exponentiel de paramètre π avant de se propager à ses deux voisins. S'ils sont occupés, ils brûlent, sinon le feu s'éteint. On étudie l'asymptotique des feux rares c'est à dire lorsque $\lambda \to 0$ et $\pi \to \infty$. On montre qu'il y a trois catégories possibles de limites d'échelles, selon le régime dans lequel λ tend vers 0 et π vers l'infini.

Abstract

Consider the following forest-fire model where the possible locations of trees are the sites of $\mathbb{Z}$. Each site has three possible states: 'vacant', 'occupied' or 'burning'. Vacant sites become occupied at rate 1. At each site, ignition (by lightning) occurs at rate λ. When a site is ignited, a fire starts and propagates to neighbors at rate π. We study the asymptotic behavior of this process as $\lambda \to 0$ and $\pi \to \infty$. We show that there are three possible classes of scaling limits, according to the regime in which $\lambda \to 0$ and $\pi \to \infty$.

II.1. Introduction

This section is devoted to preliminaries. We first define the $(\lambda,\pi)-$forest fire process with non instantaneous propagation. We next give heuristic scales and relevant quantities. Finally, we give the plan of the present chapter.

II.1.1. The discrete model

Here we introduce the forest fire model with non instantaneous propagation.

Definition II.1.1. *Let $\lambda \in (0\,,1]$ and $\pi \geq 1$ be fixed. For each $i \in \mathbb{Z}$, we consider three Poisson processes, $N^S(i) = (N_t^S(i))_{t\geq 0}, N^M(i) = (N_t^M(i))_{t\geq 0}$ and $N^P(i) = (N_t^P(i))_{t\geq 0}$ with respective parameters $1, \lambda$ and π, all of these processes being independent. Consider a $\{0,1,2\}$-valued process $(\eta_t^{\lambda,\pi}(i))_{t\geq 0, i\in\mathbb{Z}}$ such that a.s., for all $i \in \mathbb{Z}$, $(\eta_t^{\lambda,\pi}(i))_{t\geq 0}$ is càdlàg. We say that $(\eta_t^{\lambda,\pi}(i))_{t\geq 0, i\in\mathbb{Z}}$ is a $(\lambda,\pi)-$forest fire process ($(\lambda,\pi)-$FFP in short) if a.s., for all $i \in \mathbb{Z}$, all $t \geq 0$,*

$$
\begin{aligned}
\eta_t^{\lambda,\pi}(i) = &\int_0^t \boldsymbol{1}_{\{\eta_{s-}^{\lambda,\pi}(i)=0\}}\, \mathrm{d}N_s^S(i) + \int_0^t \boldsymbol{1}_{\{\eta_{s-}^{\lambda,\pi}(i)=1\}}\, \mathrm{d}N_s^M(i) \\
&+ \int_0^t \boldsymbol{1}_{\{\eta_{s-}^{\lambda,\pi}(i+1)=2, \eta_{s-}^{\lambda,\pi}(i)=1\}}\, \mathrm{d}N_s^P(i+1) + \int_0^t \boldsymbol{1}_{\{\eta_{s-}^{\lambda,\pi}(i-1)=2, \eta_{s-}^{\lambda,\pi}(i)=1\}}\, \mathrm{d}N_s^P(i-1) \\
&- 2\int_0^t \boldsymbol{1}_{\{\eta_{s-}^{\lambda,\pi}(i)=2\}}\, \mathrm{d}N_s^P(i).
\end{aligned}
$$

Formally, we say that $\eta_t^{\lambda,\pi}(i) = 0$ if there is no tree at site i at time t and $\eta_t^{\lambda,\pi}(i) = 1$ if the site i is occupied. The case $\eta_t^{\lambda,\pi}(i) = 2$ means that the site i is burning. Thus, the forest fire process starts from an empty initial configuration, seeds fall according to some i.i.d. Poisson processes of parameter 1 and matches fall according to some i.i.d. Poisson processes of parameter λ. When a seed falls on an empty site, a tree appears immediately. When a match falls on an occupied site, a fire starts and waits for an exponential time of parameter π before it propagates to its neighbors and vanishes. If its right (resp. left) neighbor is occupied then it becomes burning. Seeds falling on occupied sites, matches falling on vacant sites and fires propagating to vacant sites have no effect.

This process can be shown to exist and to be unique (for almost every realization of N^S, N^M, N^P) by using a *graphical construction.* Indeed, to build the process until a given time $T > 0$, it suffices to work between sites i which are vacant until time T [because $N_T^S(i) = 0$]. Interaction cannot cross such sites. Since such sites are a.s. infinitely many, this allows us to handle a graphical construction. It should be pointed out that this construction only works in dimension 1.

For $a, b \in \mathbb{Z}$, we set $[\![a\,,b]\!] = \{a,\dots,b\} \subset \mathbb{Z}$. For $\eta \in \{0,1,2\}^{\mathbb{Z}}$ and $i \in \mathbb{Z}$, we define the occupied connected component around i as

$$
C(\eta,i) = \begin{cases} \emptyset & \text{if } \eta(i) = 0 \text{ or } 2, \\ [\![l(\eta,i)\,, r(\eta,i)]\!] & \text{if } \eta(i) = 1, \end{cases}
$$

where $l(\eta, i) = \sup\{k < i : \eta(k) = 0 \text{ or } 2\} + 1$ and $r(\eta, i) = \inf\{k > i : \eta(k) = 0 \text{ or } 2\} - 1$.

II.1.2. Notation

In the whole paper, we use the convention $1/\infty = 0$ and $1/0 = \infty$.

We denote, for $J = [a, b]$ an interval of $\mathbb{R}$, by $|J| = b - a$ the length of J and for $\alpha > 0$, we set $\alpha J = [\alpha a, \alpha b]$.

For $I \subset \mathbb{Z}$, $|I| = \#I$ stands for the number of elements in I. For $I = [\![a\,, b]\!] = \{a, \dots, b\} \subset \mathbb{Z}$ and $\alpha > 0$, we will set $\alpha I := [\alpha a\,, \alpha b] \subset \mathbb{R}$. For $\alpha > 0$, we of course take the convention that $\alpha\emptyset = \emptyset$.

For $x \in \mathbb{R}$, $\lfloor x \rfloor$ stands for the integer part of x.

We denote by $\mathcal{I} = \{[a, b], a \leq b\}$ the set of all closed finite intervals of $\mathbb{R}$. For two intervals $[a, b]$ and $[c, d]$, we set

$$\boldsymbol{\delta}([a, b], [c, d]) = |a - c| + |b - d|, \quad \boldsymbol{\delta}([a, b], \emptyset) = |b - a|.$$

For $(x, I), (y, J)$ in $\mathbb{D}([0, T], \mathbb{R}_+ \times \mathcal{I} \cup \{\emptyset\})$, the set of càdlàg functions from $[0, T]$ into $\mathbb{R}_+ \times \mathcal{I} \cup \{\emptyset\}$, we define

$$\mathbf{d}_T((x, I), (y, J)) = \int_0^T \Big[|x(t) - y(t)| + \boldsymbol{\delta}(I_t, J_t)\Big]\, \mathrm{d}t.$$

For two functions $I, J \colon [0\,, T] \to \mathcal{I} \cup \{\emptyset\}$, we set

$$\boldsymbol{\delta}_T(I, J) = \int_0^T \boldsymbol{\delta}(I_t, J_t)\, \mathrm{d}t.$$

For $(x, t) \in \mathbb{R} \times [0\,, T]$ we also set, for $p \geq 0$,

$$\Lambda^p_{(x,t)} := \{(x + z, t - p\,|z|) : |z| \leq t/p\}$$

($(r, v) \in \Lambda^p_{(x,t)} \iff v = t - p\,|r - x|$) and its part which joins (y, s) to (x, t)

$$\Lambda^p_{(x,t)}(y, s) = \begin{cases} \{(z, t - p\,|z - x|) : z \in [x\,, y]\} & \text{if } (y, s) \in \Lambda^p_{(x,s)} \text{ and } y > x, \\ \{(z, t - p\,|z - x|) : z \in [y\,, x]\} & \text{if } (y, s) \in \Lambda^p_{(x,s)} \text{ and } y < x, \\ \emptyset & \text{else.} \end{cases}$$

Similarly, we define

$$\mathcal{V}^p_{(x,t)} = \{(x + z, t + p\,|z|) : z \in \mathbb{R}\}$$

$$\mathcal{V}^p_{(x,t)}(y, s) = \begin{cases} \{(z, t + p\,|z - x|) : z \in [x\,, y]\} & \text{if } (y, s) \in \mathcal{V}^p_{(x,t)} \text{ and } y > x, \\ \{(z, t + p\,|z - x|) : z \in [y\,, x]\} & \text{if } (y, s) \in \mathcal{V}^p_{(x,t)} \text{ and } y < x, \\ \emptyset & \text{else,} \end{cases}$$

see Figure II.1. Observe that $\Lambda^p_{(x,t)}(y, s) = \mathcal{V}^p_{(y,s)}(x, t)$. Also observe that

$$\Lambda^0_{(x,t)} = \mathcal{V}^0_{(x,t)} = \{(z, t) : z \in \mathbb{R}\} = \mathbb{R} \times \{t\}.$$

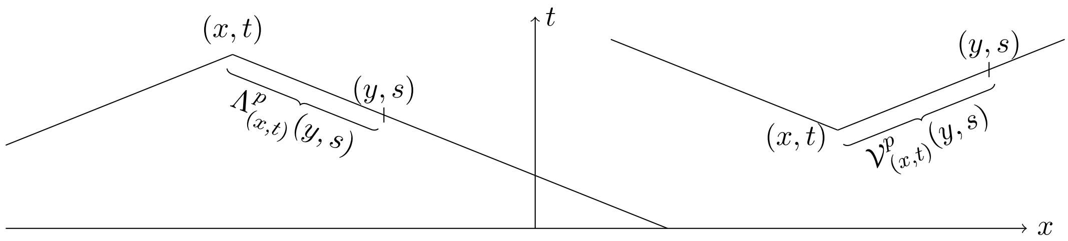

Figure II.1.: Λ^p and $\mathcal{V}^p$

On the left side is drawn $\Lambda^p_{(x,t)}$ and $\Lambda^p_{(x,t)}(y,s)$. On the right side is drawn $\mathcal{V}^p_{(x,t)}$ and $\mathcal{V}^p_{(x,t)}(y,s)$.

II.1.3. Heuristic scales and relevant quantities

We look for some time scale for which tree clusters see about one fire per unit of time. But for λ very small, clusters will be very large before a match falls inside. We thus also have to rescale space. Since we negelct fires, these quantities does not depend on π. Hence, these scales are the same as in [BF10]. We also have to find the different regimes at which $\lambda \to 0$ and $\pi \to \infty$.

Time scale

For $\lambda > 0$ very small and for t not too large, one might neglect fires, so that roughly, each site is vacant with probability e^{-t}. Indeed, the time we have to wait for the first seed follows, on each site, the law $\mathcal{E}(1)$. Thus $C(\eta_t^{\lambda,\pi},0) \simeq [\![-X\,,Y]\!]$, where X,Y are geometric random variables with parameter e^{-t}. Consequently, for t not too large,

$$\left|C(\eta_t^{\lambda,\pi},0)\right| \simeq e^t.$$

On the other hand, the rate that at which matches fall in the cluster $C(\eta_t^{\lambda,\pi},0)$ is $\lambda|C(\eta_t^{\lambda,\pi},0)|$. So we decide to accelerate time by a factor

$$\mathbf{a}_\lambda = \log(1/\lambda). \tag{II.1.1}$$

In this way, $\lambda|C(\eta_{\mathbf{a}_\lambda}^{\lambda,\pi},0)| \simeq 1$.

Space scale

We now rescale space in such a way that during a time interval of order $\mathbf{a}_\lambda = \log(1/\lambda)$, something like one match falls per unit of (space) length. Since fires occur at rate λ, our space scale has to be of order

$$\mathbf{n}_\lambda = \left\lfloor \frac{1}{\lambda \mathbf{a}_\lambda} \right\rfloor = \left\lfloor \frac{1}{\lambda \log(1/\lambda)} \right\rfloor. \tag{II.1.2}$$

This means that we will identify $[\![0\,,\mathbf{n}_\lambda]\!] \subset \mathbb{Z}$ with $[0,1] \subset \mathbb{R}$.

Rescaled clusters

We thus set, for $\lambda \in (0,1)$, $\pi \geq 1$, $t \geq 0$ and $x \in \mathbb{R}$, recalling Subsection II.1.2,

$$D_t^{\lambda,\pi}(x) := \frac{1}{\mathbf{n}_\lambda} C\left(\eta_{\mathbf{a}_\lambda t}^{\lambda,\pi}, \lfloor \mathbf{n}_\lambda x \rfloor\right). \tag{II.1.3}$$

However, this creates an immediate difficulty: recalling that $C(\eta_t^{\lambda,\pi}, 0) \simeq e^t$ for t not too large, we see that for each site x, $|D_t^{\lambda,\pi}(x)| \simeq \lambda \log(1/\lambda) e^{t\log(1/\lambda)} = \lambda^{1-t}\log(1/\lambda)$, of which the limit when $\lambda \to 0$ is 0 for $t < 1$ and $+\infty$ for $t \geq 1$.

For $t \geq 1$, there might be fires in effect and one hopes that this will make the possible limit of $|D_t^{\lambda,\pi}(x)|$ finite. However, fires can only reduce the size of clusters so that for $t < 1$, the limit of $|D_t^{\lambda,\pi}(x)|$ will really be 0. This cannot be a Markov process because it remains at 0 during a time interval of length exactly 1. We thus need to keep track of more information in order to control when it exits from 0.

To have an idea of the sizes of microscopic clusters, we keep some information about *the degree of smallness* of microscopic clusters. We consider

$$\mathbf{m}_\lambda = \left\lfloor \frac{1}{\lambda \mathbf{a}_\lambda^2} \right\rfloor = \left\lfloor \frac{1}{\lambda \log^2(1/\lambda)} \right\rfloor. \tag{II.1.4}$$

Remark that $\mathbf{m}_\lambda \ll \mathbf{n}_\lambda$ but $\mathbf{m}_\lambda \gg \lambda^{-t}$, for all $t \in [0\,,1)$. We introduce, for $\lambda > 0$, $\pi \geq 1$, $x \in \mathbb{R}$, $t \geq 0$,

$$K_t^{\lambda,\pi}(x) = \frac{\left|\left\{ i \in [\![\lfloor \mathbf{n}_\lambda x \rfloor - \mathbf{m}_\lambda\,, \lfloor \mathbf{n}_\lambda x \rfloor + \mathbf{m}_\lambda]\!] : \eta_{\mathbf{a}_\lambda t}^{\lambda,\pi}(i) = 1 \right\}\right|}{2\mathbf{m}_\lambda + 1} \in [0\,,1], \tag{II.1.5}$$

$$Z_t^{\lambda,\pi}(x) = \frac{-\log(1 - K_t^{\lambda,\pi}(x))}{\log(1/\lambda)} \wedge 1 \in [0\,,1]. \tag{II.1.6}$$

Observe that $K_t^{\lambda,\pi}(x)$ stands for the *local density of occupied sites* around $\lfloor \mathbf{n}_\lambda x \rfloor$ at time $\mathbf{a}_\lambda t$. This density is *local* because $\mathbf{m}_\lambda \ll \mathbf{n}_\lambda$. We hope that for $t < 1$, neglecting fires,

$$K_t^{\lambda,\pi}(x) \simeq 1 - \lambda^t,$$

whence $Z_t^{\lambda,\pi}(x) \simeq t$.

For all $\lambda > 0$ small enough (we need that $2\mathbf{m}_\lambda + 1 < 1/\lambda$), it also holds that $Z_t^{\lambda,\pi}(x) = 1$ if and only if $K_t^{\lambda,\pi}(x) = 1$, i.e. if and only if all the sites are occupied around $\lfloor \mathbf{n}_\lambda x \rfloor$. Indeed, $Z_t^{\lambda,\pi}(x) = 1$ implies that

$$-\log(1 - K_t^{\lambda,\pi}(x)) \geq \log(1/\lambda),$$

so that $K_t^{\lambda,\pi}(x) \geq 1 - \lambda > 1 - 1/(2\mathbf{m}_\lambda + 1)$, whence $K_t^{\lambda,\pi}(x) = 1$.

Final description

We will study the $(\lambda,\pi)-$FFP through $(D_t^{\lambda,\pi}(x), Z_t^{\lambda,\pi}(x))_{t\geq 0, x\in\mathbb{R}}$. The main idea is that for $\lambda > 0$ very small and $\pi \geq 1$ large enough:

- if $Z_t^{\lambda,\pi}(x) = z \in (0,1)$, then $|D_t^{\lambda,\pi}(x)| \simeq 0$ and the (rescaled) cluster containing x is microscopic (in the sense that the non-rescaled cluster containing $\lfloor \mathbf{n}_\lambda x \rfloor$ is small when compared to $\mathbf{n}_\lambda$), but we control the local density of occupied sites around x, which resembles $1 - \lambda^z$. Observe that this density tends to 1 as $\lambda \to 0$ for all $z \in (0,1)$;

- if $Z_t^{\lambda,\pi}(x) = 1$ and $D_t^{\lambda,\pi}(x) = [a,b]$, then the (rescaled) cluster containing x is macroscopic and has a length equal to $|b-a|$ (or $|C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}, \lfloor \mathbf{n}_\lambda x \rfloor)| \simeq \mathbf{n}_\lambda\, |b-a|$ in the original scales).

Propagation velocity

The time needed for a fire to destroy a macroscopic cluster (which contains about $\mathbf{n}_\lambda$ sites) is of order $\frac{\mathbf{n}_\lambda}{\pi}$. Indeed, a burning tree waits for an exponential time of parameter π before it propagates to neighbors. Thus, if a fire starts at 0, neglecting all other phenomena, it needs roughly a time $\mathbf{n}_\lambda/\pi$ to reach $\mathbf{n}_\lambda$. We have to compare the propagation time $\mathbf{n}_\lambda/\pi$ to the characteristic time $\mathbf{a}_\lambda$. Thus we decide to separate the three following regimes, as $\lambda \to 0$ and $\pi \to \infty$ (observe that $\frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} \simeq \frac{1}{\lambda \log^2(1/\lambda)\pi}$):

- $\frac{1}{\lambda \log^2(1/\lambda)\pi} \to 0$, which corresponds to the case where fires propagate very fast;

- $\frac{1}{\lambda \log^2(1/\lambda)\pi} \to p$, for some $p \in (0\,,\infty)$, which is an intermediate case;

- $\frac{1}{\lambda \log^2(1/\lambda)\pi} \to \infty$, which corresponds to the case where fires propagate very slowly.

Recall that, when neglecting fires and for $t < 1$, $1/\lambda^t$ is the order of magnitude of the occupied cluster around 0 at time $\mathbf{a}_\lambda t$. Thus a match falling in 0 at time $\mathbf{a}_\lambda t$ needs a time of order $1/(\lambda^t \pi)$ to destroy the whole component. In order to treat the last case, we suppose that there exists $z_0 \in [0\,,1)$ such that

$$\frac{1}{\lambda^t \pi} \to \begin{cases} 0 & \text{if } t < z_0, \\ \infty & \text{if } t > z_0. \end{cases} \tag{II.1.7}$$

This means that if the match falls at time $\mathbf{a}_\lambda t < \mathbf{a}_\lambda z_0$, there are few occupied sites around 0. Thus the fire destroys the whole component in a time of order $1/(\lambda^t \pi) \ll \mathbf{a}_\lambda$. On the other hand, if the match falls a time $\mathbf{a}_\lambda t > \mathbf{a}_\lambda z_0$ then the component is too big to be destroyed before $\mathbf{a}_\lambda T$, for all $T > 0$.

To summarize, we will treat separately the three following regimes, as $\lambda \to 0$ and $\pi \to \infty$.

1. $\mathcal{R}(0)$: $\frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} \ll 1$, the fast regime;

2. $\mathcal{R}(p)$: $\frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} \sim p \in (0\,,\infty)$, the intermediate regime;

3. $\mathcal{R}(\infty, z_0)$: $\frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} \gg 1$ and $\frac{\log(\pi)}{\log(1/\lambda)} \to z_0 \in [0\,,1]$, the slow regime.

Definition II.1.2. *Let (E, d) be a metric space.*

Let $p \geq 0$. In the rest of the paper, we will say that $f(\lambda, \pi) \in E$ tends to $\ell \in E$ when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$ if for all $\delta > 0$, there are $\varepsilon > 0$ and $\lambda_0 \in (0\,,1]$ such that for all $\lambda \in (0\,,\lambda_0)$ and all $\pi \geq 1$ in such a way that $\left|\frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} - p\right| < \varepsilon$, there holds $d(f(\lambda, \pi), \ell) < \delta$.

Let $z_0 \in [0\,,1]$. Similarly, we will say that $f(\lambda, \pi) \in E$ tends to $\ell \in E$ when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(\infty, z_0)$ if for all $\delta > 0$, there are $\varepsilon > 0$, $K_0 > 0$ and $\lambda_0 \in (0\,,1]$ such that for all $\lambda \in (0\,,\lambda_0)$ and all $\pi \geq 1$ in such a way that $\frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} \geq K_0$ and $\left|\frac{\log(\pi)}{\log(1/\lambda)} - z_0\right| < \varepsilon$, there holds $d(f(\lambda, \pi), \ell) < \delta$.

II.1.4. Plan of the chapter

In Section II.2, we give our main results (scaling limits and cluster-size distribution) together with heuristic proof. In Section II.3, we study the existence and uniqueness of the limit process. In Section II.4, we study the effect of fires in the discrete process, which will be usefull in the rest of the chapter (propagation through an occupied zone). In Section II.5, we give a discrete version of Section II.3. The rest of the chapter is devoted to the rigorous proof of our results: we treat the convergence in the regime $\mathcal{R}(\infty, z_0)$ in Section II.7, in the regime $\mathcal{R}(p)$, for some $p \in (0\,,\infty)$ in Section II.8 and finally in the regime $\mathcal{R}(0)$ in Section II.9. In the end of each two last sections, we deduce estimates on the cluster size distribution for the process.

II.2. Main results

II.2.1. Main results when $p \in [0\,,\infty)$

In this section, we are interested in the regime $\mathcal{R}(p)$, for some $p \in [0\,,\infty)$. We treat together the cases $p = 0$ and $p \in (0\,,\infty)$. There are just few differences between these two cases: see Remark II.2.2 for an alternative definition in the case $p = 0$.

II.2.1.1. Definition of the limit forest fire process

We now describe the limit process. We want this process to be Markov and this forces us to add some variables. We consider a Poisson measure $\pi_M(\mathrm{d}x, \mathrm{d}t)$ on $\mathbb{R} \times [0, \infty)$, with intensity measure $\mathrm{d}x\,\mathrm{d}t$, whose marks correspond to matches. We use Notation II.1.2.

Definition II.2.1. *Let $p \geq 0$. A process $(Z_t(x), H_t(x), F_t(x))_{t\geq 0, x\in\mathbb{R}}$ with values in $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{N}$ such that a.s., for all $x \in \mathbb{R}$, $(Z_t(x), H_t(x))_{t\geq 0}$ is càdlàg, is said to be a $p-$limit-forest-fire-process (or LFFP(p) in short), if a.s., for all $t \geq 0$, all $x \in \mathbb{R}$,*

$$\begin{aligned}
Z_t(x) &= \int_0^t \mathbf{1}_{\{Z_s(x)<1\}}\,\mathrm{d}s - \sum_{s\leq t}(F_s(x) \wedge 1),\\
H_t(x) &= \int_0^t Z_{s-}(x)\mathbf{1}_{\{Z_{s-}(x)<1\}}\pi_M(\{x\} \times \mathrm{d}s) - \int_0^t \mathbf{1}_{\{H_s(x)>0\}}\,\mathrm{d}s, \qquad \text{(II.2.1)}\\
F_t(x) &= \iint_{(y,s)\in\Lambda^p_{(x,t)}} \mathbf{1}_{\{\forall (r,v)\in\Lambda^p_{(x,t)}(y,s)\,,\,Z_{v-}(r)=1 \text{ and } H_{v-}(r)=0\}}\pi_M(\mathrm{d}y, \mathrm{d}s).
\end{aligned}$$

To the LFFP(p), we associate the process $D_t(x) = [L_t(x), R_t(x)]$, with

$$\begin{aligned}
L_t(x) &= \sup\{y \leq x :\ Z_t(y) < 1 \text{ or } H_t(y) > 0\},\\
R_t(x) &= \inf\{y \geq x :\ Z_t(y) < 1 \text{ or } H_t(y) > 0\}.
\end{aligned}$$

A typical path of the finite box version of the LFFP(p) is drawn and commented in Figure II.3 and a simulation algorithm is explained in the proof of Proposition II.3.4.

Remark II.2.2. *If $p = 0$, we can rewrite the process $(Z_t(x), H_t(x), F_t(x))_{t\geq 0, x\in\mathbb{R}}$ as follow*

$$\begin{aligned}
Z_t(x) &= \int_0^t \mathbf{1}_{\{Z_s(x)<1\}}\,\mathrm{d}s - \int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{Z_{s-}(x)=1, y\in D_{s-}(x)\}}\pi_M(\mathrm{d}y, \mathrm{d}s),\\
H_t(x) &= \int_0^t Z_{s-}(x)\mathbf{1}_{\{Z_{s-}(x)<1\}}\pi_M(\{x\} \times \mathrm{d}s) - \int_0^t \mathbf{1}_{\{H_s(x)>0\}}\,\mathrm{d}s,\\
F_t(x) &= \int_{\mathbb{R}} \mathbf{1}_{\{Z_{t-}(x)=1, y\in D_{t-}(x)\}}\pi_M(\mathrm{d}y \times \{t\}),
\end{aligned}$$

where $D_{t-}(x)$ is defined as above. Indeed, for all $x \in \mathbb{R}$, all $t \geq 0$,

$$\Big\{(y, s) : \forall (r, v) \in \Lambda^0_{(x,t)}(y, s) : Z_{v-}(r) = 1 \text{ and } H_{v-}(r) = 0\Big\} = D_t(x) \times \{t\}$$

With a slightly different formulation, this limit process is the same as in [BF10] where the propagation is instantaneous. This relationship is very natural. Indeed, the case $p = 0$ corresponds to the case where the propagation velocity is very high.

II.2.1.2. Formal dynamics

Let us explain the dynamics of this process. For $p \in [0, \infty)$, we consider $T > 0$ fixed and set $\mathcal{A}_T = \{x \in \mathbb{R} : \ \pi_M(\{x\} \times [0, T]) > 0\}$. For each $t \geq 0$, $x \in \mathbb{R}$, $D_t(x)$ stands for the occupied cluster containing x. We call this cluster *microscopic* if $D_t(x) = \{x\}$. Otherwise, we call it *macroscopic*.

1. Initial condition. We have $Z_0(x) = H_0(x) = F_0(x) = 0$ and $D_0(x) = \{x\}$ for all $x \in \mathbb{R}$.

2. Occupation of vacant zones. We consider here $x \in \mathbb{R} \setminus \mathcal{A}_T$. Then we have $H_t(x) = 0$ for all $t \in [0, T]$. When $Z_t(x) < 1$, $D_t(x) = \{x\}$ and $Z_t(x)$ stands for the *local density of occupied sites* around x. Then $Z_t(x)$ grows linearly until it reaches 1, as described by the first term on the RHS of the first equation in (II.2.1). When $Z_t(x) = 1$, the cluster containing x is macroscopic and is described by $D_t(x)$.

3. Microscopic fires. Here we assume that $x \in \mathcal{A}_T$ and that the corresponding mark of π_M happens at some time t where $Z_{t-}(x) < 1$. In such a case, the cluster containing x is microscopic. Then we set $H_t(x) = Z_{t-}(x)$, as described by the first term on the RHS of the second equation of (II.2.1) and we leave unchanged the value of $Z_t(x)$ and $F_t(x)$. We then let $H_t(x)$ decrease linearly until it reaches 0, see the second term on the RHS of the second equation in (II.2.1). At all times where $H_t(x) > 0$, that is during $[t, t + Z_{t-}(x))$, the site x acts like a barrier (see Point 4. below).

4. Macroscopic fires. Here we assume that $y \in \mathcal{A}_T$ and that the corresponding mark of π_M happens at some time s where $Z_{s-}(y) = 1$. This means that the cluster containing y is macroscopic. Thus this mark creates 2 fires: one goes to the left, the other to the right. These fires propagates along of $\mathcal{V}^p_{(y,s)}$, until they are stopped by a microscopic zone or a barrier or an other fire.

In other words, for all $(x, t) \in \mathbb{R} \times \mathbb{R}_+$, we set $F_t(x) = 0$ unless there exists one (or two) mark (y, s) of π_M such that $(y, s) \in \Lambda^p_{(x,t)}$ (or equivalently $(x, t) \in \mathcal{V}^p_{(y,s)}$) and for all $(r, v) \in \Lambda^p_{(x,t)}(y, s), Z_{v-}(r) = 1$ and $H_{v-}(r) = 0$, in which case we set $F_t(x) = 1$ (or $F_t(x) = 2$). When x is crossed by a fire, $Z_t(x)$ jumps from 1 to 0, see the second term on the RHS of the first equation in (II.2.1).

5. Clusters. Finally the definition of the clusters $(D_t(x))_{x\in\mathbb{R}}$ becomes more clear: these clusters are delimited by zones with local density smaller than 1 (i.e. $Z_t(y) < 1$) or by sites where a microscopic fire has (recently) started (i.e. $H_t(y) > 0$).

II.2.1.3. Well posedness

The existence and uniqueness of the LFFP(0) has been proved in [BF10]. The proof in the case $p \in (0, \infty)$ is in the same spirit.

Theorem II.2.3. *For any Poisson measure $\pi_M(\mathrm{d}x, \mathrm{d}t)$ on $\mathbb{R} \times [0, \infty)$ with intensity measure $\mathrm{d}x\,\mathrm{d}t$, there a.s. exists a unique LFFP(p). Furthermore, it can be constructed graphically and its restriction to any finite box $[0, T] \times [-n, n]$ can be perfectly simulated.*

The LFFP(p) $(Z_t(x), H_t(x), F_t(x))_{t\geq 0, x\in\mathbb{R}}$ is furthermore Markov, since it solves a well-posed time homogeneous Poisson-driven S.D.E.

II.2.1.4. The convergence result

Theorem II.2.4. *Consider for each $\lambda \in (0, 1], \pi \geq 1$, the process $(Z_t^{\lambda,\pi}(x), D_t^{\lambda,\pi})_{t\geq 0, x\in\mathbb{R}}$ associated to the $(\lambda, \pi)-$FFP. Consider also the LFFP(p) $(Z_t(x), H_t(x), F_t(x))_{t\geq 0, x\in\mathbb{R}}$ and the associated $(D_t(x))_{t\geq 0, x\in\mathbb{R}}$. We assume that $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$, for some $p \in [0, \infty)$.*

1. *For any $T > 0$, any finite subset $\{x_1, \ldots, x_q\} \subset \mathbb{R}$,*
$$(Z_t^{\lambda,\pi}(x_i), D_t^{\lambda,\pi}(x_i))_{t\in[0,T], i=1,\ldots,q}$$
goes in law to $(Z_t(x_i), D_t(x_i))_{t\in[0,T], i=1,\ldots,q}$ in $\mathbb{D}([0, T], \mathbb{R} \times (\mathcal{I} \cup \{\emptyset\}))$. Here the space $\mathbb{D}([0, T], \mathbb{R} \times (\mathcal{I} \cup \{\emptyset\}))$ is endowed with the distance $\mathbf{d}_T$.

2. *For any finite subset $\{(x_1, t_1), \ldots, (x_q, t_q)\} \subset \mathbb{R}\times[0, \infty)$, $(Z_{t_i}^{\lambda,\pi}(x_i), D_{t_i}^{\lambda,\pi}(x_i))_{i=1,\ldots,q}$ goes in law to $(Z_{t_i}(x_i), D_{t_i}(x_i))_{i=1,\ldots,q}$ in $(\mathbb{R} \times (\mathcal{I} \cup \{\emptyset\}))^q$. Here $\mathcal{I} \cup \{\emptyset\}$ is endowed with $\boldsymbol{\delta}$.*

3. *For all $t > 0$,*
$$\left(\frac{\log(|C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}, 0)|)}{\log(1/\lambda)} \mathbf{1}_{\{|C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}, 0)| \geq 1\}}\right) \wedge 1$$
goes in law to $Z_t(0)$.

Point 3 will allow us to check some estimates on the cluster-size distribution. Since we deal with finite-dimensional marginals in space, it is quite clear that the processes H and F do not appear in the limit, since for each $x \in \mathbb{R}$, for all $t \geq 0$, a.s., $H_t(x) = F_t(x) = 0$. (of course, it is false that a.s., for all $x \in \mathbb{R}$, all $t \geq 0, H_t(x) = F_t(x) = 0$). We obtain the convergence of $D^{\lambda,\pi}$ (resp. $Z^{\lambda,\pi}$) to D (resp. Z) only when integrating in time. We cannot hope for a Skorokhod convergence since the limit process $D(x)$ (resp. $Z(x)$) jumps instantaneously from $\{x\}$ (resp. 1) to some interval with positive length (resp. 0), while $D^{\lambda,\pi}(x)$ (resp. $Z^{\lambda,\pi}(x)$) needs many small jumps, in a very short interval, to become macroscopic (resp. empty).

The space $(\mathbb{D}([0, T], \mathbb{R} \times (\mathcal{I} \cup \{\emptyset\})), \mathbf{d}_T)$ is not a complete metric space since $\mathbf{d}_T$ is too weak. However, it seems that it is not really a problem because in the proof, we use a coupling argument and obtain a convergence in probability.

II.2.1.5. Heuristics argument

We now explain roughly the reasons why Theorem II.2.4 holds. We consider a $(\lambda,\pi)-$FFP $(\eta_t^{\lambda,\pi}(i))_{t\geq 0,i\in\mathbb{Z}}$ and the associated process $(Z_t^{\lambda,\pi}(x),\ D_t^{\lambda,\pi}(x))_{t\geq 0,x\in\mathbb{R}}$. We assume below that λ is very small, π very large and $\mathbf{n}_\lambda/(\mathbf{a}_\lambda\pi)$ close to p.

0. Scales. With our scales, there are $\mathbf{n}_\lambda=\lfloor 1/(\lambda\log(1/\lambda))\rfloor$ sites per unit of length. Approximately one fire starts per unit of time per unit of length. A vacant site becomes occupied at rate $\mathbf{a}_\lambda=\log(1/\lambda)$.

1. Initial condition. We have, for all $x\in\mathbb{R}$, $(Z_0^{\lambda,\pi}(x),D_0^{\lambda,\pi}(x))=(0,\emptyset)\simeq(0,\{x\})$.

2. Occupation of vacant zones. Assume that no match falls in a zone $[a,b]$ (which correspond to the zone $[\![\mathbf{n}_\lambda a\,,\mathbf{n}_\lambda b]\!]$ before rescaling) during $[0\,,1]$ (or $[0\,,\mathbf{a}_\lambda]$ before rescaling).

a. For $s\in[0,1)$, we have
$$D_s^{\lambda,\pi}(x)\simeq[x\pm\lambda^{1-s}]\simeq\{x\}$$
and $Z_s^{\lambda,\pi}(x)\simeq s$ for all $x\in[a,b]$.

Indeed, each site is occupied with probability $1-e^{-\mathbf{a}_\lambda s}=1-\lambda^s$. Thus the local density is roughly $K_t^{\lambda,\pi}\simeq 1-\lambda^s$, whence $Z_t^{\lambda,\pi}(x)\simeq s$, while the typical size of occupied clusters is λ^s, whence $D_s^{\lambda,\pi}(x)\simeq[x\pm\lambda^s/\mathbf{n}_\lambda]\simeq[x\pm\lambda^{1-s}]$.

b. At time $s=1$, $Z_1^{\lambda,\pi}(x)\simeq 1$ and all the sites in $[a,b]$ are occupied (with very high probability).

Indeed, we have $(b-a)\mathbf{n}_\lambda$ sites and each of them is occupied at time 1 with probability $1-e^{-\mathbf{a}_\lambda}=1-\lambda$ so that all of them are occupied with probability $(1-\lambda)^{(b-a)\mathbf{n}_\lambda}\simeq e^{-(b-a)/\log(1/\lambda)}$, which goes to 1 as $\lambda\to 0$.

Assume now that the zone around x (i.e. the zone $[\![\lfloor\mathbf{n}_\lambda x\rfloor-\mathbf{m}_\lambda\,,\lfloor\mathbf{n}_\lambda x\rfloor+\mathbf{m}_\lambda]\!]$ before rescaling) has been destroyed at time t (or at time $\mathbf{a}_\lambda t$ before rescaling) by a fire. Then, observations 2a. and 2b. above still hold:

i. for $s\in[0\,,1)$ and if no fire starts in $[\![\lfloor\mathbf{n}_\lambda x\rfloor-\mathbf{m}_\lambda\,,\lfloor\mathbf{n}_\lambda x\rfloor+\mathbf{m}_\lambda]\!]$ during $[\mathbf{a}_\lambda t\,,\mathbf{a}_\lambda(t+s)]$, we have $D_{t+s}^{\lambda,\pi}(x)\simeq[x\pm\lambda^{1-s}]\simeq\{x\}$ and $Z_{t+s}^{\lambda,\pi}(x)\simeq s$;

ii. $Z_{t+1}^{\lambda,\pi}(x)\simeq 1$ and all the sites around x are occupied at time $t+1$ with very high probability.

3. Microscopic fires. Assume that a fire starts at some location x (i.e. $\lfloor\mathbf{n}_\lambda x\rfloor$ before rescaling) at some time t (or $\mathbf{a}_\lambda t$ before rescaling) with $Z_{t-}^{\lambda,\pi}(x)=z\in(0,1)$. The possible clusters on the left and right of x cannot be connected during (approximately) $[t,t+z]$, but they can be connected after (approximately) $t+z$. In other words, x acts like a barrier during $[t,t+z]$.

Indeed, the connected component A of x (or $\lfloor\mathbf{n}_\lambda x\rfloor$ before rescaling) at time t (or $\mathbf{a}_\lambda t$ before rescaling) has a size of order λ^{1-z} (which thus contains approximately $\lambda^{1-z}\mathbf{n}_\lambda\simeq$

λ^{-z} sites). The fire destroys the component A in a time of order $1/(\lambda^z \mathbf{a}_\lambda \pi) \ll 1$ (or $1/(\lambda^z \pi) \ll \mathbf{a}_\lambda$ in original scale). Thus this fire crosses very fast the component A and each site of A becomes burning and then empty (i.e. $\eta^{\lambda,\pi}(i)$ jumps from 1 to 2 then from 2 to 0) during the time interval $[t\,, t+1/(\lambda^z \mathbf{a}_\lambda \pi)] \simeq \{t\}$ (or $[\mathbf{a}_\lambda t\,, \mathbf{a}_\lambda t + 1/(\lambda^z \pi)] \simeq \{\mathbf{a}_\lambda t\}$ before rescaling). The probability that a fire starts again in A is very small. Thus, using the same computation as in point 2, we observe that $\mathbb{P}$[A is completely occupied at time $t+s$]$\simeq (1-\lambda^s)^{\lambda^{-z}} \simeq e^{-\lambda^{s-z}}$. When $\lambda \to 0$, this quantity tends to 0 if $s < z$ and to 1 if $s > z$.

4. Macroscopic fires. Assume, now, that a fire starts at some place x (i.e. $\lfloor \mathbf{n}_\lambda x \rfloor$ before rescaling) at some time t (or $\mathbf{a}_\lambda t$ before rescaling) and that $Z_{t-}^{\lambda,\pi}(x) \simeq 1$. Thus, $D_{t-}^{\lambda,\pi}(x)$ is macroscopic (i.e. its length is of order 1 in our scales). Then the match creates two fires: one propagates to the left and one to the right at speed p (p unit times per unit space). There are only two burning trees at each instant with very high probability. Of course, these fires are stopped when they meet a vacant site (i.e. a microscopic zone or a barrier) or another fire.

Indeed, we have to wait for an exponential time of parameter π between each propagation in the original scales. It then produces two independent Poisson processes of parameter π which stand for the location of the fires. Then, for $b > x$, this Poisson process is at $\lfloor \mathbf{n}_\lambda b \rfloor$ in the original scale (or in b after rescaling) roughly at time $\mathbf{a}_\lambda t + (\mathbf{n}_\lambda/\pi)(b-x)$ (or at time $t + (\mathbf{n}_\lambda/(\mathbf{a}_\lambda \pi))(b-x) \simeq t + p(b-x)$ after rescaling). All sites $i \in [\![\lfloor \mathbf{n}_\lambda x \rfloor\,, \lfloor \mathbf{n}_\lambda b \rfloor]\!]$ becomes successively burning and empty roughly at time $\mathbf{a}_\lambda t + (i - \lfloor \mathbf{n}_\lambda x \rfloor)/\pi$ in the original scale (or the site $y = i/\mathbf{n}_\lambda \in \mathbb{R}$ is burning at time $t + p(y-x)$ after rescaling).

5. Clusters. For $t \geq 0$, $x \in \mathbb{R}$, the cluster $D_t^{\lambda,\pi}(x)$ resembles $[x \pm \lambda^{1-z}] \simeq \{x\}$ if $Z_t^{\lambda,\pi}(x) = z \in (0,1)$. We then say that x is microscopic. Now, macroscopic clusters are delimited either by microscopic zones or by sites where there has been recently a microscopic fire (see point 3) or by a burning tree.

Comparing the arguments above to the rough description of the LFFP(p) (see Section II.2.1.2), our hope is that the (λ,π)-FFP resembles the LFFP(p) for $\lambda > 0$ very small, π very large and $1/(\lambda {\mathbf{a}_\lambda}^2 \pi)$ close to p.

Remark II.2.5. *Remark II.2.2 is now more clear. Consider the regime $\mathcal{R}(0)$. If a fire starts at x (or $\lfloor \mathbf{n}_\lambda x \rfloor$ before rescaling) at time t (or $\mathbf{a}_\lambda t$ before rescaling), the time needed to reach a point b (or $\lfloor \mathbf{n}_\lambda b \rfloor$ before rescaling) is roughly $\mathbf{n}_\lambda |b-x|/(\mathbf{a}_\lambda \pi) \simeq 0$ (or $\mathbf{n}_\lambda (b-x)/\pi \ll \mathbf{a}_\lambda$ before rescaling). It means that if $b \in D_{t-}^0(x)$ (or $\lfloor \mathbf{n}_\lambda b \rfloor \in C(\eta_{\mathbf{a}_\lambda t-}^{\lambda,\pi}, \lfloor \mathbf{n}_\lambda x \rfloor)$ before rescaling) the fire reaches b at time $t + \mathbf{n}_\lambda |b-x|/(\mathbf{a}_\lambda \pi) \simeq t$. In the scaling limit, the cluster containing x is thus destroyed instantaneously.*

II.2.1.6. Cluster size distribution

We will deduce from Theorem II.2.4 the following estimates on the cluster-size distribution.

Corollary II.2.6. *Let $p \in [0, \infty)$ be fixed. Let $(Z_t(x), H_t(x), F_t(x))_{t\geq 0, x\in\mathbb{R}}$ be a LFFP(p) and $(D_t(x))_{t\geq 0,x\in\mathbb{R}}$ the associated process. For each $\lambda \in (0,1]$ and $\pi \geq 1$, let $(\eta_t^{\lambda,\pi}(i))_{t\geq 0,i\in\mathbb{Z}}$ be a $(\lambda,\pi)-$FFP.*

(a). For all $t \geq (5+p)/2$, all $0 < a < b < 1$, for some $0 < c_1 < c_2$ depending on p, as $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$,

$$\lim_{\lambda,\pi} \mathbb{P}\left[\left|C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}, 0)\right| \in [1/\lambda^a\,, 1/\lambda^b]\right] = \mathbb{P}\left[Z_t(0) \in [a\,, b]\right] \in [c_1(b-a)\,, c_2(b-a)].$$

(b). For all $t \geq 3/2$, all $B > 0$, for some $0 < c_1 < c_2$ and $0 < \kappa_1 < \kappa_2$ depending on p, as $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$,

$$\lim_{\lambda,\pi} \mathbb{P}\left[\left|C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}, 0)\right| \geq B\mathbf{n}_\lambda\right] = \mathbb{P}\left[|D_t(0)| \geq B\right] \in [c_1 e^{-\kappa_2 B}, c_2 e^{-\kappa_1 B}].$$

This result shows that there is a *phase transition* around the critical size $\mathbf{n}_\lambda$: the cluster-size distribution changes of shape at $\mathbf{n}_\lambda$. The main idea is that two types of clusters are present: macroscopic clusters, of which the size is of order $\mathbf{n}_\lambda$ and microscopic clusters, of which the size is smaller than $\mathbf{n}_\lambda$.

II.2.2. Main results for $p = \infty$

In this section, we are interested in the regime $\mathcal{R}(\infty, z_0)$, for some $z_0 \in [0, 1]$.

II.2.2.1. Definition of the limit process

In this regime, the limit process is much simpler, in the sense that fires only have a local (in space) effect (but can have long time effect). This is due to the fact that a fire can't go too far away in a finite time.

We consider a Poisson measure $\pi_M(\mathrm{d}x, \mathrm{d}t)$ on $\mathbb{R}\times[0,\infty)$, with intensity measure $\mathrm{d}x\,\mathrm{d}t$, whose marks correspond to matches.

Definition II.2.7. *Let $z_0 \in [0, 1]$. A process $(Y_t(x))_{t\geq 0,x\in\mathbb{R}}$ with values in $\mathbb{R}_+$ such that a.s., for all $x \in \mathbb{R}$, $(Y_t(x))_{t\geq 0}$ is càdlàg, is said to be a LFFP(∞, z_0) if a.s., for all $t \geq 0$, all $x \in \mathbb{R}$,*

$$Y_t(x) = \int_0^{t\wedge z_0} s\,\pi_M(\{x\}\times \mathrm{d}s) - \int_0^t \mathbf{1}_{\{Y_s(x)\in[0,1)\}}\,\mathrm{d}s + \mathbf{1}_{\{t\geq z_0\}}\pi_M(\{x\}\times[z_0\,, t]). \quad \text{(II.2.2)}$$

The process Y takes its values in $[0, 1]$ and can be non-zero only at locations where $\pi_M(\{x\}\times\mathbb{R}) \neq 0$. If the mark of π_M happens at time $t < z_0$, then the (microscopic) cluster containing x is destroyed instantaneously and $Y_s(x) \in (0, 1)$ during $[t\,, 2t)$: x acts like a barrier during this time interval. If the mark happens at time $t > z_0$ then the cluster containing x is too big to be destroyed and $Y_s(x) = 1$ for ever: there is always a burning tree close to x. We then naturally associate the process $D_t(x) = [L_t(x)\,, R_t(x)]$, with

$$L_t(x) = \begin{cases} x & \text{if } t < 1, \\ \sup\{y \le x :\ Y_t(y) > 0\} & \text{if } t \ge 1; \end{cases}$$

$$R_t(x) = \begin{cases} x & \text{if } t < 1, \\ \inf\{y \ge x :\ Y_t(y) > 0\} & \text{if } t \ge 1. \end{cases}$$

A typical path of the finite box version of the LFFP(∞, z_0) is drawn and commented in Figure II.2.

Remark II.2.8. *The process Y is a time inhomogeneous Markov process. To make it homogeneous, we can add a second variable Z as in the first equation* (II.2.1) *in the Definition II.2.1.*

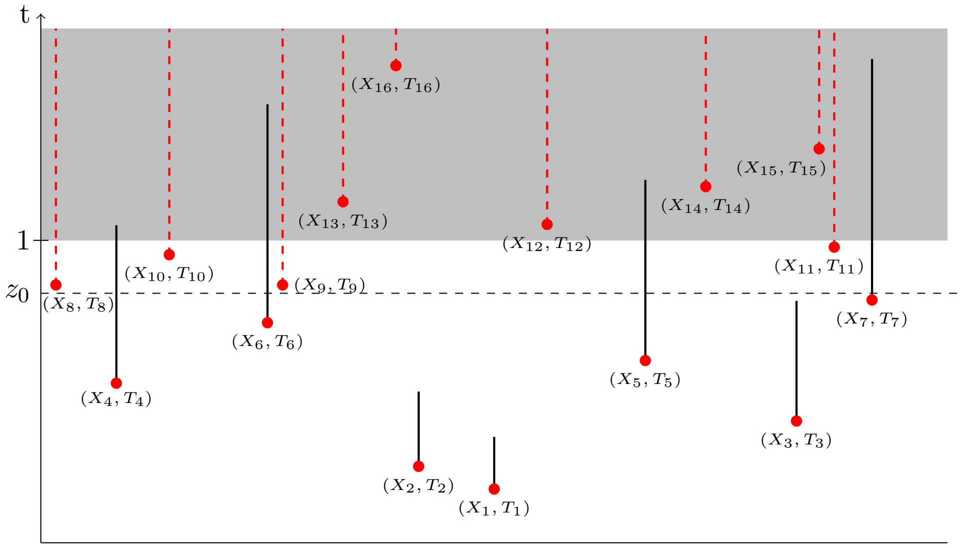

Figure II.2.: LFF$(\infty, z_0)-$process in a finite box.

The marks of π_M are represented by $\bullet$'s. The filled zones represents zones in which $|D(x)| > 0$. The plain vertical segments represent the sites where $Y_t(x) \in (0\,,1)$ and the dashed vertical segments represent the sites where $Y_t(x) = 1$. In the rest of the space, we always have $Y_t(x) = 0$. Until time 1, all the particles are microscopic. Matches 1 to 7 falls before z_0. At each of these marks, a process Y starts and its life-time equals the instant where it has started. This creates a barrier with height T_k (the segment above T_k ends at time $2T_k$). The other matches falls after z_0. At each of these marks, a process Y starts and remains equal to 1 forever.
Thus, for each $x \in [-A\,,A]$, $D_t^A(x) = \{x\}$ for $t \in [0\,,1)$ and merge at $t = 1$. Here we have at time 1 the clusters $[-A, X_8]$, $[X_8, X_4]$, $[X_4, X_{10}]$, $[X_{10}, X_6]$, $[X_6, X_9]$, $[X_9, X_5]$, $[X_5, X_{11}]$, $[X_{11}, X_7]$ and , $[X_7, A]$.
Remark that $t \mapsto |D_t(x)|$ is non-increasing on $[2z_0\,,\infty)$ for all x.

II.2.2.2. Formal dynamics

Let us explain the dynamics of this process. We consider $\mathcal{A} = \{x \in \mathbb{R} : \pi_M(\{x\} \times [0,\infty)) > 0\}$. For each $t \geq 0$, $x \in \mathbb{R}$, $D_t(x)$ stands for the occupied cluster containing x. We call this cluster *microscopic* if $D_t(x) = \{x\}$. Otherwise, we call it *macroscopic*.

1. Initial condition. We have $Y_0(x) = 0$ and $D_0(x) = \{x\}$ for all $x \in \mathbb{R}$.

2. Occupation of vacant zones. We consider here $x \in \mathbb{R} \setminus \mathcal{A}$. Then we have $Y_t(x) = 0$ for all $t \in [0,\infty)$. When $t < 1$, $D_t(x) = \{x\}$. When $t \geq 1$, the cluster containing x is macroscopic and is described by $D_t(x)$.

3. First kind of fires. Here we assume that $x \in \mathcal{A}$ and that the corresponding mark of π_M happens at some time $t < z_0$. We set $Y_t(x) = t$, as described by the first term on the RHS of the equation of (II.2.2). We then let $Y_t(x)$ decrease linearly until it reaches 0, see the second term on the RHS of the equation in (II.2.2) (i.e. $Y_s(x) = \min(2t - s, 0)\mathbf{1}_{\{s \geq t\}}$).

4. Second kind of fires. Here we assume that $x \in \mathcal{A}$ and that the corresponding mark of π_M happens at some time t where $t > z_0$. Then we set $Y_s(x) = 1$ for all $s \in [t,\infty)$ see the third term of the RHS of the equation (II.2.2).

5. Clusters. Finally the definition of the clusters $(D_t(x))_{x \in \mathbb{R}}$ becomes more clear: these clusters remain microscopic until $t = 1$. For $t \geq 1$, $(D_t(x))_{x \in \mathbb{R}, t \geq 1}$ is delimited by sites where a fire of first kind has (recently) started (i.e. $Y_t(y) \in (0,1)$) or by sites where a fire of second kind has started (i.e. $Y_t(y) = 1$). Remark that for $t \geq 2z_0$, only fires of second kind delimit the clusters.

II.2.2.3. Well posedness

The following proposition is obvious from the definition, see Figure II.2.

Proposition II.2.9. *Let π_M be a Poisson measure on $\mathbb{R} \times [0,\infty)$ with intensity measure $\mathrm{d}x\,\mathrm{d}t$. There a.s. exists a unique LFFP(∞, z_0) $(Y_t(x))_{t \geq 0, x \in \mathbb{R}}$. It can be simulated exactly on any finite box $[0,T] \times [-n,n]$.*

II.2.2.4. The convergence result

We will prove the following result.

Theorem II.2.10. *Let $z_0 \in [0,1]$. Consider for each $\lambda \in (0,1]$ and $\pi \geq 1$ the process $(D_t^{\lambda,\pi}(x))_{t \geq 0, x \in \mathbb{R}}$ associated with the $(\lambda,\pi)-$FFP. Consider also the LFFP(∞, z_0) $(Y_t(x))_{t \geq 0, x \in \mathbb{R}}$ and the associated $(D_t(x))_{t \geq 0, x \in \mathbb{R}}$ process. We assume that $\lambda \to 0$ and $\pi \to \infty$ in the slow regime $\mathcal{R}(\infty, z_0)$.*

1. *For any $T > 0$, any finite subset $\{x_1, \ldots, x_q\} \subset \mathbb{R}$, $(D_t^{\lambda,\pi}(x_i))_{t \in [0,T], i=1,\ldots,q}$ goes in law to $(D_t(x_i))_{t \in [0,T], i=1,\ldots,q}$ in $\mathbb{D}([0,T], \mathcal{I})^q$. Here $\mathbb{D}([0,T], \mathcal{I})^q$ is endowed with $\boldsymbol{\delta}_T$.*
2. *For any finite subset $\{(x_1,t_1), \ldots, (x_q,t_q)\} \subset \mathbb{R} \times [0,\infty)$, $(D_{t_i}^{\lambda,\pi}(x_i))_{i=1,\ldots,q}$ goes in law to $(D_{t_i}(x_i))_{t \in [0,T], i=1,\ldots,q}$ in $\mathcal{I}^q$, $\mathcal{I}$ being endowed with $\boldsymbol{\delta}$.*

II.2.2.5. Heuristics arguments

We assume below that $\lambda > 0$ is very small, $\pi \geq 1$ is very large, $\lambda \mathbf{a}_\lambda^2 \pi$ is close to 0 and $\log(\pi)/\log(1/\lambda)$ is close to z_0.

0. Scales. With our scales, there are $\mathbf{n}_\lambda = \lfloor 1/(\lambda \log(1/\lambda)) \rfloor$ sites per unit of length. Approximately one fire starts per unit of time per unit of length. A vacant site becomes occupied at rate $\mathbf{a}_\lambda = \log(1/\lambda)$.

1. Initial condition. We have, for all $x \in \mathbb{R}$, $D_0^{\lambda,\pi}(x) = \emptyset \simeq \{x\}$ and $D_0(x) = \{x\}$.

2. Occupation of vacant zones. Exactly as in the regime $\mathcal{R}(p)$, $D_t^{\lambda,\pi}(x) \simeq [x \pm \lambda^{1-t}] \simeq \{x\}$ for $t < 1$ and the clusters become macroscopic at time 1.

3. First kind of fires. Assume that a match falls at some place x (or $\lfloor \mathbf{n}_\lambda x \rfloor$ in the original scales) at some time $t < z_0$ (or $\mathbf{a}_\lambda t < \mathbf{a}_\lambda z_0$ in the original scales). Then the fire burns almost immediately the occupied cluster and it needs roughly a time t (or $\mathbf{a}_\lambda t$ in the original scales) to be filled again. Thus x acts like a barrier during $[t\,, 2t)$.

Indeed, the connected component A of x (or $\lfloor \mathbf{n}_\lambda x \rfloor$ before rescaling) at time t (or $\mathbf{a}_\lambda t$ before rescaling) has a size of order λ^{1-t} (which thus contains approximately $\lambda^{1-t}\mathbf{n}_\lambda \simeq \lambda^{-t}$ sites). The fire destroys the component A in a time of order $1/(\lambda^t \mathbf{a}_\lambda \pi) \ll 1$ (or $1/(\lambda^t \pi) \ll \mathbf{a}_\lambda$ in original scales) due to $\mathcal{R}(\infty, z_0)$. Thus this fire crosses very fast the component A and each site of A becomes burning and then empty (i.e. $\eta^{\lambda,\pi}(i)$ jumps from 1 to 2 then from 2 to 0) during the time interval $[t\,, t + 1/(\lambda^t \mathbf{a}_\lambda \pi)] \simeq \{t\}$ (or $[\mathbf{a}_\lambda t\,, \mathbf{a}_\lambda t + 1/(\lambda^t \pi)] \simeq \{\mathbf{a}_\lambda t\}$ before rescaling). The probability that a fire starts again in A is very small. Thus, we observe that $\mathbb{P}[\text{A is completely occupied at time } t+s] \simeq (1-\lambda^s)^{\lambda^{-z}} \simeq e^{-\lambda^{s-z}}$. When $\lambda \to 0$, this quantity tends to 0 if $s < t$ and to 1 if $s > t$.

4. Second kind of fires. Assume that a match falls at some place x (or $\lfloor \mathbf{n}_\lambda x \rfloor$ in the original scales) at some time $t > z_0$ (or $\mathbf{a}_\lambda t > \mathbf{a}_\lambda z_0$ in the original scales). Then the fire needs an infinite time (in our scales) to burn the occupied cluster, so that there is a burning site close to x forever.

Indeed, $D_t^{\lambda,\pi}(x)$ contains roughly λ^{-t} sites if $t \in (z_0\,, 1)$ and $\mathbf{n}_\lambda$ sites if $t \geq 1$. In any case, the time needed for the fire to cross this cluster is of order $\left| D_t^{\lambda,\pi}(x) \right| / \pi$, which is very large when compared to $\mathbf{a}_\lambda$ in the regime $\mathcal{R}(\infty, z_0)$. Thus, the fire cannot reach the rim of $D_t^{\lambda,\pi}(x)$.

5. Clusters. For $t \geq 0$, $x \in \mathbb{R}$, the cluster $D_t^{\lambda,\pi}(x)$ resembles $[x \pm \lambda^{1-z}] \simeq \{x\}$ if $t < 1$. Now, macroscopic clusters emerge when $t \geq 1$ and are delimited either by a burning tree or by sites where there has been recently a microscopic fire (see point 3).

Comparing the arguments above to the rough description of the LFFP(∞, z_0) (see Section II.2.2.2), our hope is that the $(\lambda, \pi)-$FFP resembles the LFFP(∞, z_0) in the regime $\mathcal{R}(\infty, z_0)$.

II.2.2.6. Cluster-size distribution

The following corollary is easily deduced from the Theorem II.2.10.

Corollary II.2.11. *Let $z_0 \in [0\,,1]$. Let $(Y_t(x))_{t\geq 0, x\in\mathbb{R}}$ be a LFFP(∞, z_0) and $(D_t(x))_{t\geq 0, x\in\mathbb{R}}$ the associated process. For each $\lambda \in (0,1]$ and $\pi \geq 1$, let $(\eta_t^{\lambda,\pi}(i))_{t\geq 0, i\in\mathbb{Z}}$ be a $(\lambda,\pi)-$FFP.*

For all $t > 2z_0$, as $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(\infty, z_0)$,

$$\frac{1}{\mathbf{n}_\lambda}\left|C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}, 0)\right| \xrightarrow{\mathcal{L}} |D_t(0)| \sim \Gamma(2, t - z_0).$$

This result shows that for t large enough, there are only macroscopic clusters, that is clusters with size of order $\mathbf{n}_\lambda$.

We immediately give the proof of Corollary II.2.11. For $t \geq 0$, Theorem II.2.10 shows that, when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(\infty, z_0)$,

$$\frac{1}{\mathbf{n}_\lambda}\left|C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}, 0)\right| \xrightarrow{\mathcal{L}} |D_t(0)|\,.$$

Furthermore, if $t > 2z_0$, only fires of the second kind (i.e. matches falling after z_0) still have an effect. Indeed, when a match falls in x at time $t < z_0$, it creates a barrier in x during $[t\,, 2t) \subset [0\,, 2z_0]$. Thus, $D_t(0)$ is only delimited by sites where a match has fallen during $[z_0\,, t]$. This is a Poisson process on $\mathbb{R}$ with intensity $t - z_0$. Consequently,

$$|D_t(0)| \sim \Gamma(2, t - z_0).$$

II.2.2.7. Irreversibility

It might look surprising at the first glance that the limit process is non-reversible while the discrete process is reversible. Indeed, for $t \geq 1 \wedge 2z_0$, clusters in the limit process are macroscopic and the sizes are non-increasing. On the other hand, in the discrete process, it is quite clear that, when working in a finite box, the process returns to its original state. This is due to the time scale: we have to wait a very long time to observe again the original state.

II.3. Existence and uniqueness of the limit process

The goal of this section is to show that the limit processes are well-defined, unique, can be obtained from a graphical construction and can be restricted to a finite box.

II.3.1. Restriction of the LFFP(∞, z_0) to a finite box

Let $z_0 \in [0\,,1]$ be fixed. In this subsection, we study the LFFP(∞, z_0).

Proposition II.3.1. *Let π_M a Poisson measure on $\mathbb{R}\times[0\,,\infty)$ with intensity measure* $\mathrm{d}x\,\mathrm{d}t$ *and $A>0$.*

1. *The values of $(Y_t(x))_{t\geq 0, x\in[-A,A]}$ are entirely determined by $\pi_M|_{[-A,A]\times\mathbb{R}_+}$. Actually, for all $x\in\mathbb{R}$, the values of $(Y_t(x))_{t\geq 0}$ are entirely determined by $\pi_M|_{\{x\}\times\mathbb{R}_+}$.*

2. *There exist some constants $\alpha>0$ and $C>0$, not depending on $A>0$, such that*

$$\mathbb{P}\left[(D_t(x))_{t\geq 0, x\in[-A/2,A/2]}\subset[-A\,,A]\right]\geq 1-Ce^{-\alpha A}. \tag{II.3.1}$$

Proof. The first part of Proposition II.3.1 is obvious from the definition of the process $(Y_t(x))_{t\geq 0,x\in\mathbb{R}}$. In order to prove the second part, consider the event Ω_A^+ on which π_M has at least one mark (X_1,τ_1) in $[A/2\,,A]\times(3/4\,,1)$ and at least one mark (X_2,τ_2) in $[A/2\,,A]\times(1\,,3/2)$.

Observe now that on Ω_A^+,

$$Y_t(X_1)>0,\text{ for all } t\in[\tau_1\,,2\tau_1)\supset[1\,,3/2],$$

because X_1 is a either a fire of first kind (if $\tau_1\leq z_0$), whence $Y_t(X_1)=(2\tau_1-t)_+$ for all $t\geq\tau_1$, or X_1 is a fire of second kind (if $\tau_1>z_0$), whence $Y_t(X_1)=1$ for all $t\geq\tau_1$. Besides, X_2 is always a fire of second kind (because $\tau_2>1\geq z_0$) whence $Y_t(X_2)=1$ for all $t\in[\tau_2\,,\infty)\supset[3/2\,,\infty]$ (X_2 burns for ever).

Similarily, we define the event Ω_A^- on which π_M has at least one mark $(\tilde{X}_1,\tilde{\tau}_1)$ in $[-A\,,-A/2]\times(3/4\,,1)$ and at least one mark $(\tilde{X}_2,\tilde{\tau}_2)$ in $[-A\,,-A/2]\times(1\,,3/2)$. On Ω_A^-, there holds that

$$Y_t(\tilde{X}_1)>0,\text{ for all } t\in[1\,,3/2]\subset[\tilde{\tau}_1\,,2\tilde{\tau}_1)\text{ and } Y_t(\tilde{X}_2)=1,\text{ for all } t\geq 3/2\geq\tilde{\tau}_2.$$

Thus, on $\Omega_A^+\cap\Omega_A^-$, $D_t(x)\subset[-A\,,A]$ for all $t\geq 0$ and all $x\in[-A/2\,,A/2]$. Finally, we can bound from below the left hand side of (II.3.1) by

$$\mathbb{P}\left[\Omega_A^+\cap\Omega_A^-\right]\geq 1-2(e^{-A/8}+e^{-A/4})\geq 1-4e^{-A/8}$$

whence (II.3.1) with $C=4$ and $\alpha=1/8$. □

Definition II.3.2. *Let $z_0 \in [0\,,1]$ and $(Y_t(x))_{x\in\mathbb{R},t\geq}$ be a LFFP(∞, z_0). For all $A > 0$ and for $x \in [-A\,,A]$, we define the process $D_t^A(x) = [L_t^A(x)\,, R_t^A(x)]$, with*

$$L_t^A(x) = \begin{cases} x & \textit{if } t < 1, \\ \sup\{y \leq x : \ Y_t(y) > 0\} \vee (-A) & \textit{if } t \geq 1; \end{cases}$$
$$R_t^A(x) = \begin{cases} x & \textit{if } t < 1, \\ \inf\{y \geq x : \ Y_t(y) > 0\} \wedge \quad A & \textit{if } t \geq 1. \end{cases}$$

As a corollary of Proposition II.3.1, we have, for $A > 0$,

$$\mathbb{P}\left[(D_t(x))_{t\geq 0, x\in[-A/2,A/2]} = (D_t^A(x))_{t\geq 0,x\in[-A/2,A/2]}\right] \geq 1 - Ce^{-\alpha A}.$$

II.3.2. Restriction of the LFFP(p) to a finite box

The aim of this subsection is to prove Theorem II.2.3. We define an analogous process of LFFP(p) on a finite space interval, which can be perfectly simulated. We then show that these two processes are equal with very high probability.

II.3.2.1. Algorithm

Let $p \in [0\,,\infty)$. Here we show that when working on a finite space interval, the LFFP(p) is somewhat discrete. We consider a Poisson measure $\pi_M(\mathrm{d}x, \mathrm{d}t)$ on $\mathbb{R} \times [0\,,\infty)$ with intensity measure $\mathrm{d}x\,\mathrm{d}t$.

Definition II.3.3. *Let $A > 0$. A process $(Z_t^A(x), H_t^A(x), F_t^A(x))_{t\geq 0, x\in[-A,A]}$ with values in $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{N}-$ such that a.s., for all $x \in [-A\,,A]$, $(Z_t^A(x), H_t^A(x))_{t\geq 0}$ is càdlàg, is a $A-$LFFP(p) if a.s., for all $t \geq 0$, all $x \in [-A\,,A]$,*

$$\begin{aligned}
Z_t^A(x) &= \int_0^t \mathbf{1}_{\{Z_s^A(x)<1\}}\,\mathrm{d}s - \sum_{s\leq t}(F_s^A \wedge 1), \\
H_t^A(x) &= \int_0^t Z_{s-}^A(x)\mathbf{1}_{\{Z_{s-}^A(x)<1\}}\pi_M(\{x\}\times \mathrm{d}s) - \int_0^t \mathbf{1}_{\{H_s^A(x)>0\}}\,\mathrm{d}s, \qquad \text{(II.3.2)} \\
F_t^A(x) &= \iint_{(y,s)\in\Lambda^p_{(x,t)}\cap([-A,A]\times[0,\infty))} \mathbf{1}_{\{\forall (r,v)\in \Lambda^p_{(x,t)}(y,s),\, Z_{v-}^A(r)=1 \textit{ and } H_{v-}^A(r)=0\}}\pi_M(\mathrm{d}y, \mathrm{d}s).
\end{aligned}$$

To the $A-$LFFP(p), as usual, we associate the process $D_t^A(x) = [L_t^A(x), R_t^A(x)]$, with

$$\begin{aligned}
L_t^A(x) =& (-A) \vee \sup\{y \in [-A\,,x] : \ Z_t^A(y) < 1 \text{ or } H_t^A(y) > 0\}, \\
R_t^A(x) =& A \wedge \inf\{y \in [x\,,A] : \ Z_t^A(y) < 1 \text{ or } H_t^A(y) > 0\}.
\end{aligned}$$

A typical path of $(Z_t^A(x), H_t^A(x), F_t^A(x))_{t\geq 0, x\in[-A,A]}$ is drawn in figure II.3.
The proof of the following proposition shows the construction of the $A-$LFFP(p) in an algorithmic way.

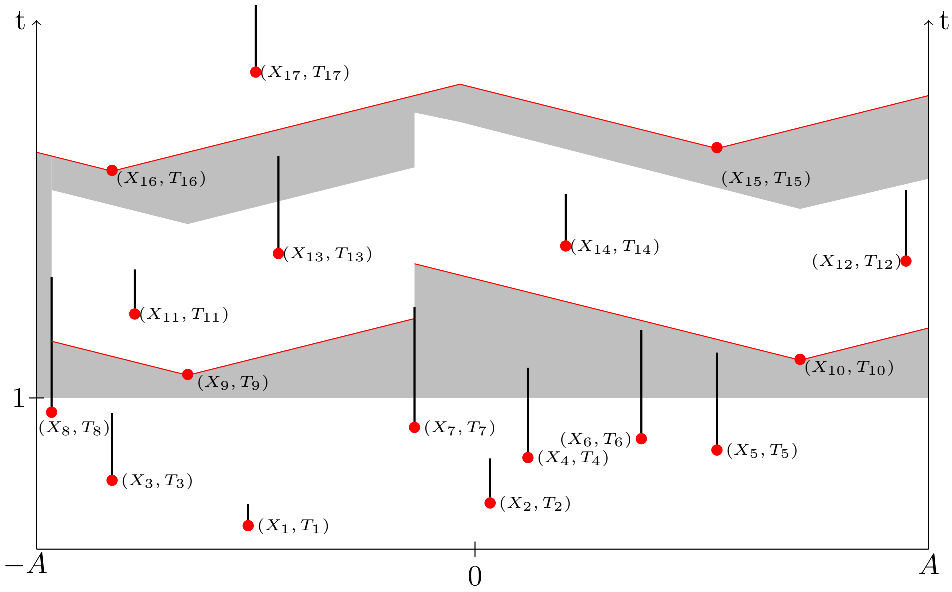

Figure II.3.: LFFP(p) in a finite box

The marks of π_M (matches) are represented as $\bullet$'s. The filled zones represent zones in which $Z_t^A(x) = 1$, that is macroscopic clusters. In the rest of the space, we always have $Z_t^A(x) < 1$. The plain vertical segments represent the sites where $H_t^A(x) > 0$. $F_t^A(x) = 0$ except on the lines with slope p where $F_t^A(x) = 1$ or $F_t^A(x) = 2$ in the crossing point of the fires starting in (X_{15}, T_{15}) and (X_{16}, T_{16}). Until time 1, all of the clusters are microscopic. The first eigth marks of the Poisson measure fall in that zone. As a consequence, at each of these marks, a process H^A starts. Their lifetime is equal to the instant where they have started (e.g., the segment above (X_1, T_1) ends at time $2T_1$). At time 1, all clusters where there has been no mark become macroscopic and merge together. However, this is limited by vertical segments. Here, at time 1, we have the clusters $[-A, X_8]$, $[X_8, X_7]$, $[X_7, X_4]$, $[X_4, X_6]$, $[X_6, X_5]$ and $[X_5, A]$. The segment above (X_4, T_4) ends at time $2T_4$ and thus, at this time, the clusters $[X_7, X_4]$ and $[X_4, X_6]$ merge into $[X_7, X_6]$. The ninth mark falls in the (macroscopic) zone $[X_8, X_7]$ and thus two fires start. They cross the cluster $[X_8, X_7]$ at speed p, i.e. cross $[X_8, X_7]$ with a slope p. A process H^A then starts at X_{11} at time T_{11}. Since $Z^A_{T_{11}-}(X_{11}) = T_{11} - (T_9 + p\,|X_9 - X_{11}|)$ [because $Z^A_{T_9+p|X_9-X_{11}|}(X_{11})$ has been set to 0], the segment above (X_{11}, T_{11}) will end at time $2T_{11} - (T_9 + p\,|X_9 - X_{11}|)$. On the other hand, a fire starts at X_{10} at time T_{10} and crosses the cluster of X_{10} at speed p. A site x in $[X_7, A]$ remains microscopic from time $T_{10} + p\,|X_{10} - x|$ until time $T_{10} + p\,|X_{10} - x| + 1$. The two matches 14 and 12 create microscopic fires (because they fall on sites where $Z_t^A(x) < 1$). Observe finally that the 15th and the 16th fires are stopped by each oher.

With this realization, we have $0 \in (X_7, X_2)$ and, thus, $Z_t^A(0) = t$ for $t \in [0, 1]$, then $Z_t^A(0) = 1$ for $t \in [1, T_{10} + pX_{10})$, then $Z_t^A(0) = t - (T_{10} + pX_{10})$ for $t \in [T_{10} + pX_{10}, T_{10} + pX_{10} + 1)$, then $Z_t^A(0) = 1$ for $t \in [T_{10} + pX_{10} + 1, T_{16} + pX_{15})$, etc. We also see that $D_t^A(0) = \{0\}$ for $t \in [0, 1)$, $D_t^A(0) = [X_7, X_4]$ for $t \in [1, 2T_4)$, $D_t^A(0) = [X_7, X_6]$ for $t \in [2T_4, 2T_6)$, $D_t^A(0) = [X_7, X_{10} + \frac{T_{10}-t}{p})$ for $t \in [2T_6, T_{10}+pX_{10})$, $D_t^A(0) = \{0\}$ for $t \in [T_{10}+pX_{10}, T_{10}+pX_{10}+1)$, etc. We finally have $F_t^A(0) = 0$ for all $t \neq \{T_{10} + pX_{10}, T_{15} + pX_{15}\}$ and $F^A_{T_{10}+pX_{10}}(0) = F^A_{T_{15}+pX_{15}}(0) = 1$.

Proposition II.3.4. *Consider a Poisson measure $\pi_M(\mathrm{d}x, \mathrm{d}t)$ on $\mathbb{R} \times [0,\infty)$ with intensity measure $\mathrm{d}x\,\mathrm{d}t$. For any $A > 0$ and $p \geq 0$, there a.s. exists a unique $A-$LFFP(p) which can be perfectly simulated.*

Algorithm. Here we only treat the case $p > 0$. The case $p = 0$ is much easier and has been treated in [BF10], as mentioned in Remark II.2.2.

Consider the marks $(X_k, T_k)_{k=1,\dots,n}$ of π_M in $[-A, A] \times [0, T]$, ordered chronologically and set $T_0 = 0$. We describe the construction via an algorithm, which also shows uniqueness, in the sense that there is no choice in the construction.

Suppose that we have built the process $(Z_t^A(x), H_t^A(x), F_t^A(x))_{x\in[-A,A]}$ at some time $t \geq 0$. We then can set

$$\begin{aligned}
\chi_t^+ &= \Big\{x \in [-A, A] : F_t^A(x) = 1 \text{ and } Z_t^A(x+) = 1\Big\},\\
\chi_t^- &= \Big\{x \in [-A, A] : F_t^A(x) = 1 \text{ and } Z_t^A(x-) = 1\Big\},\\
\chi_t^0 &= \Big\{x \in [-A, A] : H_t^A(x) > 0 \text{ or } Z_t^A(x+) \neq Z_t^A(x-)\Big\} \cup \{-A, A\},\\
\chi_t &= \chi_t^+ \cup \chi_t^- \cup \chi_t^0,
\end{aligned}$$

where $Z_t^A(x+) = \lim_{y\to x, y>x} Z_t^A(y)$ (resp. $Z_t^A(x-) = \lim_{y\to x, y<x} Z_t^A(y)$). Observe that χ_t^+ (resp. χ_t^-) is the set of fires at time t that spread to the right (resp. to the left) and that χ_t^0 is the set of sites where a fire can be stopped (barrier or microscopic zone). We also define, for $r > t$,

$$\mathcal{E}_t^r := \bigcup_{x\in\chi_t^+, y\in\chi_t^-} \mathcal{V}_{(x,t)}^p \cap \mathcal{V}_{(y,t)}^p \cap ([-A, A] \times [t, r]) \tag{II.3.3}$$

$$\cup \bigcup_{x\in\chi_t^+\cup\chi_t^-, y\in\chi_t^0} \mathcal{V}_{(x,t)}^p \cap (\{y\} \times [t, r]). \tag{II.3.4}$$

The set (II.3.3) is the possible locations (y, s) where two fires may meet during $[t, r]$. The set (II.3.4) is the possible locations (y, s) where a fire may be stopped by a microscopic zone or a barrier during $[t, r]$. Thus, $\mathcal{E}_t^r$ is the set of possible locations (y, s) where a fire may be stopped during $[t, r]$, when no match falls in $[-A, A]$ during $[t, r]$.

Step 0. Put $Z_0^A(x) = H_0^A(x) = F_0^A(x) = 0$ for all $x \in [-A, A]$.

Assume that, for some $q \in \{0, \dots, n-1\}$, the process $(Z_t^A(x), H_t^A(x), F_t^A(x))_{t\in[0,T_q],x\in[-A,A]}$ has been built.

Step $q+1$. We build $(Z_t^A(x), H_t^A(x), F_t^A(x))_{t\in(T_q,T_{q+1}],x\in[-A,A]}$ in the following way: for $x \in [-A, A]$ and $t \in (T_q, T_{q+1})$, we set $H_t^A(x) = \max(0, H_{T_q}^A(x) - (t - T_q))$. We then set, recall (II.3.3) and (II.3.4),

$$\mathcal{E}_{T_q}^{T_{q+1}} = \Big\{(X_q^1, T_q^1), \dots, (X_q^N, T_q^N)\Big\}$$

ordered chronologically, and put $(X_q^0, T_q^0) = (X_q, T_q)$ and $(X_q^{N+1}, T_q^{N+1}) = (X_{q+1}, T_{q+1})$. Observe that a.s. $T_q = T_q^0 < T_q^1 < \cdots < T_q^N < T_q^{N+1} = T_{q+1}$. Assume that the process has been built until T_q^k, for some $k \in \{0, \dots, N\}$. We then build the process on $(T_q^k, T_q^{k+1}]$. Recall that no match falls in $[-A, A]$ during the time interval (T_q^k, T_q^{k+1}).

We first compute $(F_t^A(x))_{t\in(T_q^k,T_q^{k+1}),x\in[-A,A]}$. Since a fire can't be stopped during (T_q^k, T_q^{k+1}), if $x \in \chi^+_{T_q^k}$, we set $F_s^A(y) = 1$ for all $(y,s) \in \mathcal{V}^p_{(x,T_q^k)}(x + \frac{T_q^{k+1}-T_q^k}{p}, T_q^{k+1})$, recall Subsection II.1.2, while, if $x \in \chi^-_{T_q^k}$, we set $F_s^A(y) = 1$ for all $(y,s) \in \mathcal{V}^p_{(x,T_q^k)}(x - \frac{T_q^{k+1}-T_q^k}{p}, T_q^{k+1})$. Otherwise, that is if $(y,s) \notin \left(\bigcup_{x\in\chi^+_{T_q^k}} \mathcal{V}^p_{(x,T_q^k)}(x + \frac{T_q^{k+1}-T_q^k}{p}, T_q^{k+1})\right) \cup \left(\bigcup_{x\in\chi^-_{T_q^k}} \mathcal{V}^p_{(x,T_q^k)}(x - \frac{T_q^{k+1}-T_q^k}{p}, T_q^{k+1})\right)$, we set $F_s^A(y) = 0$. To summarize, for all $(y,s) \in [-A, A] \times (T_q^k, T_q^{k+1})$, we have

$$F_s^A(y) = \begin{cases} 1 & \text{if } y - \frac{s-T_q^k}{p} \in \chi^+_{T_q^k} \\ 1 & \text{if } y + \frac{s-T_q^k}{p} \in \chi^-_{T_q^k} \\ 0 & \text{else.} \end{cases}$$

We then compute $(Z_t^A(x))_{t\in(T_q^k,T_q^{k+1}),x\in[-A,A]}$. Let us fix $x \in [-A, A]$. We set $N_x := \#\left\{s \in (T_q^k, T_q^{k+1}) : F_s^A(x) = 1\right\}$ and $\tau_0 := T_q^k$. If $N_x \geq 1$, for $j = 0, \dots, N_x - 1$, we set $\tau_{j+1} := \inf\left\{s \in (\tau_j, T_q^{k+1}) : F_s^A(x) = 1\right\}$). While x isn't crossed by a fire, $Z_s^A(x)$ grows linearly. We thus have, for all $s \in (T_q^k, T_q^{k+1})$

$$Z_s^A(x) = \begin{cases} \min(Z^A_{T_q^k}(x) + s - T_q^k, 1) & \text{if } s \in (T_q^k, \tau_1), \\ \min(s - \tau_j, 1) & \text{if } s \in [\tau_j, \tau_{j+1}) \text{ and } N_x \geq j \geq 1, \\ \min(s - \tau_{N_x}, 1) & \text{if } s \in [\tau_{N_x}, T_q^{k+1}). \end{cases}$$

if $N_x \geq 1$, whereas

$$Z_s^A(x) = \min(Z^A_{T_q^k}(x) + s - T_q^k, 1)$$

if $N_x = 0$.

We finally compute $F^A_{T_q^{k+1}}(x)$, $Z^A_{T_q^{k+1}}(x)$ and $H^A_{T_{q+1}}(x)$ for all $x \in [-A, A]$.

Case 1. If $x \neq X_q^{k+1}$, observe that at most one fire can reach x at time T_q^{k+1} (else $x \in \mathcal{E}^{T_q^{k+1}}_{T_q^k}$). If $x - \frac{T_q^{k+1}-T_q^k}{p} \in \chi^+_{T_q^k}$ or $x + \frac{T_q^{k+1}-T_q^k}{p} \in \chi^-_{T_q^k}$, that is if a fire reaches x at time T_q^{k+1}, we set $F^A_{T_q^{k+1}}(x) = 1$ and $Z^A_{T_q^{k+1}}(x) = 0$. Else, we set $F^A_{T_q^{k+1}}(x) = 0$ and $Z^A_{T_q^{k+1}}(x) = Z^A_{T_q^{k+1}-}(x)$.

Case 2. If $x = X_q^{k+1}$ and $k < N$, observe that X_q^{k+1} isn't crossed by a fire during (T_q^k, T_q^{k+1}) i.e. $N_{X_q^{k+1}} = 0$. If $X_q^{k+1} - \frac{T_q^{k+1}-T_q^k}{p} \notin \chi^+_{T_q^k}$ and $X_q^{k+1} + \frac{T_q^{k+1}-T_q^k}{p} \notin \chi^-_{T_q^k}$ (i.e. if the fire which might have reached X_q^{k+1} has been stopped before T_q^k) or if

$H^A_{T_q^{k+1}-}(X_q^{k+1}) > 0$ or $Z^A_{T_q^{k+1}-}(X_q^{k+1}) < 1$ (i.e. if there has been recently a microscopic fire), then put $F^A_{T_q^{k+1}}(X_q^{k+1}) = 0$. Else, there is one (or two) fire that reaches X_q^{k+1} at time T_q^{k+1} and we set $F^A_{T_q^{k+1}}(X_q^{k+1}) = 1$ (or 2). To summarize, we put

$$F^A_{T_q^{k+1}}(X_q^{k+1}) = \mathbf{1}_{\{H^A_{T_q^{k+1}-}(X_q^{k+1})=0 \text{ and } Z^A_{T_q^{k+1}-}(X_q^{k+1})=1\}} \times \left(\mathbf{1}_{\{X_q^{k+1} - \frac{T_q^{k+1}-T_q^k}{p} \in \chi^+_{T_q^k}\}} + \mathbf{1}_{\{X_q^{k+1} + \frac{T_q^{k+1}-T_q^k}{p} \in \chi^-_{T_q^k}\}} \right).$$

We finally put

$$Z^A_{T_q^{k+1}}(X_q^{k+1}) = Z^A_{T_q^{k+1}-}(X_q^{k+1}) \mathbf{1}_{\{F^A_{T_q^{k+1}}(X_q^{k+1})=0\}}.$$

Case 3. If $x = X_{q+1} = X_q^{N+1}$ and $k = N$, a match falls in X_{q+1} at time $T_{q+1} = T_q^{N+1}$. We then set

$$Z^A_{T_{q+1}}(X_{q+1}) = Z^A_{T_{q+1}-}(X_{q+1}) \mathbf{1}_{\{Z^A_{T_{q+1}-}(X_{q+1})<1\}}$$

and

$$F^A_{T_{q+1}}(X_{q+1}) = \mathbf{1}_{\{Z^A_{T_{q+1}-}(X_{q+1})=1\}}.$$

To conclude the construction, we set, for all $x \in [-A\,,A]$

$$H^A_{T_{q+1}}(x) = \begin{cases} H^A_{T_{q+1}-}(x) & \text{if } x \neq X_{q+1}, \\ Z^A_{T_{q+1}-}(X_{q+1}) \mathbf{1}_{\{Z^A_{T_{q+1}-}(X_{q+1})<1\}} & \text{if } x = X_{q+1}. \end{cases} \qquad \square$$

II.3.2.2. Restriction of the LFFP(p) to a finite box

We now prove a refined version of Theorem II.2.3.

Proposition II.3.5. *Let $p \in [0\,,\infty)$ and π_M be a Poisson measure on $\mathbb{R} \times [0\,,\infty)$ with intensity measure* $\mathrm{d}x\,\mathrm{d}t$.

1. *There exists a unique LFFP(p) $(Z_t(x), H_t(x), F_t(x))_{t\geq 0, x\in\mathbb{R}}$.*
2. *It can be perfectly simulated on $[-n\,,n] \times [0\,,T]$ for any $T > 0$, any $n > 0$.*
3. *For $A > 0$, let $(Z^A_t(x), H^A_t(x), F^A_t(x))_{t\geq 0, x\in[-A,A]}$ be the unique $A-$LFFP(p) and the associated $(D^A_t(x))_{t\geq 0, x\in[-A,A]}$. There holds*

$$\mathbb{P}\Big[(Z_t(x), H_t(x), F_t(x), D_t(x))_{t\in[0,T], x\in[-A/2,A/2]} = (Z^A_t(x), H^A_t(x), F^A_t(x), D^A_t(x))_{t\in[0,T], x\in[-A/2,A/2]}\Big] \geq 1 - C_T e^{-\alpha_T A} \quad \text{(II.3.5)}$$

for some constants $\alpha_T > 0$ and $C_T > 0$ not depending on $A > 0$.

Proof. We divide the proof into several step. We work on $[0\,,T]$.

Step 1. We observe that for a mark (X,τ) of π_M with $X \in [-A\,,A]$, we have $H_t^A(X) > 0$ or $Z_t^A(X) < 1$ for all $t \in [\tau\,,\tau + 1/2)$.

Indeed, assume first that $Z_{\tau-}^A(X) \in [0\,,1/2)$. Then $Z_t^A(X) = Z_{\tau-}^A(X) + t - \tau < 1$ for all $t \in [\tau\,,\tau + 1/2]$.

Assume next that $Z_{\tau-}^A(X) \in [1/2\,,1)$. Then $H_\tau^A(X) = Z_{\tau-}^A \geq 1/2$, so that $H_t^A(X) = H_\tau^A(X) - t + \tau > 0$ for all $t \in [\tau\,,\tau + 1/2)$.

If finally $Z_{\tau-}^A(X) = 1$, then $Z_\tau^A(X) = 0$, whence $Z_t^A(X) = t - \tau < 1$ for $t \in [\tau\,,\tau + 1)$.

Step 2. For $a \in \mathbb{R}$, we consider the event Ω_a^l defined as follows: for $\{(X_k, T_k)\}_{k=1,\dots,n}$ the marks of π_M restricted to $[a\,,a+1) \times [0\,,T]$ ordered chronologically, for $T_0 = 0$, $T_{n+1} = T$, we put

$$\Omega_a^l = \left\{ \max_{i=0,\dots,n} (T_{i+1} - T_i) < 1/4 \right\} \cap \left\{ \min_{i=1,\dots,n-1} (X_{i+1} - X_i) > 0 \right\}.$$

We immediately deduce from Step 1 that for any $a \in \mathbb{R}$, any $A > |a| + 1$,

$$\Omega_a^l \subset \{\exists x : [0\,,T] \to (a\,,a+1), t \mapsto x_t \text{ non decreasing} \\ \text{and for all } t \in [0\,,T], H_t^A(x_t) > 0 \text{ or } Z_t^A(x_t) < 1\}.$$

Thus, on Ω_a^l, clusters on the left of a cannot be connected to clusters on the right of $a + 1$ during $[0\,,T]$. Furthermore, since the function x is non decreasing, a fire starting from the left of a can't cross the zone $(a\,,a+1)$ (i.e. it necessarily would be stopped by some x_{t_0}). Thus, matches falling at the left of a do not affect the zone $(a+1\,,\infty)$.

In the same way, we put $\Omega_a^r = \{\max_{i=0,\dots,n}(T_{i+1} - T_i) < 1/4\} \cap \{\max_{i=1,\dots,n-1}(X_{i+1} - X_i) < 0\}$. We of course have, for any $a \in \mathbb{R}$, $A > |a| + 1$,

$$\Omega_a^r \subset \{\exists y : [0\,,T] \to (a\,,a+1), t \mapsto y_t \text{ non increasing} \\ \text{and for all } t \in [0\,,T], H_t^A(y_t) > 0 \text{ or } Z_t^A(y_t) < 1\}.$$

As above, on Ω_a^r, clusters on the right of $a+1$ cannot be connected to clusters on the left of a during $[0\,,T]$ and the fact that y is non increasing ensures us that matches falling on the right on $a + 1$ do not affect the zone $(-\infty\,,a)$.

Step 3. Obviously, $q_T = \mathbb{P}\left[\Omega_a^l\right] = \mathbb{P}\left[\Omega_a^r\right]$ is positive and does not depend on a. Furthermore, Ω_a^l (resp. Ω_a^r) is independent of Ω_b^l (resp. Ω_b^r) for all $a, b \in \mathbb{Z}$ with $a \neq b$. Hence there are a.s. infinitely many $a \in \mathbb{Z}$ (resp. $b \in \mathbb{Z}$) such that Ω_a^l (resp. Ω_b^r) is realized.

Then it is routine to deduce the well-posedness of the LFFP(p). The perfect simulation algorithm on a finite-box $[-n\,,n] \times [0\,,T]$ is also easy: find $a_1 < a_2$ with $a_1 + 1 < -n < n < a_2$ such that $\Omega_{a_1}^l \cap \Omega_{a_2}^r$ is realized. Then apply the same rules as for the $A-$LFFP(p) to simulate the process in $[a_1\,,a_2+1]$. This will give the true LFFP(p) inside $[a_1+1\,,a_2]$ during $[0\,,T]$.

Finally, we can clearly bound from below the left hand side of (II.3.5) by

$$\mathbb{P}\left[(\cup_{a\in[-A,-A/2-1]\cap\mathbb{Z}}\,\Omega_a^l) \cap (\cup_{a\in[A/2,A-1]\cap\mathbb{Z}}\,\Omega_a^r)\right] \geq 1 - 2(1 - q_T)^{A/2-2}$$

whence (II.3.5) with $C_T = 2/(1-q_T)^2$ and $\alpha_T = -\log(1-q_T)/2$. $\square$

II.4. Propagation Lemmas

Here we study the propagation of a fire through an occupied cluster. When a match falls on an occupied cluster, two fires start: one goes to the left and one goes to the right. This propagation is not necessarily linear, it sometimes can regress. However there are few 'sparks'.

Consider two families of Poisson processes $(N_t^S(i))_{t\geq 0, i\in\mathbb{Z}}$ and $(N_t^P(i))_{t\geq 0, i\in\mathbb{Z}}$ with respective rates 1 and π, all these processes being independent. We consider *the propagation process ignited at* $(0,0)$ defined by

$$\begin{aligned}\check{\zeta}_t^{\lambda,\pi}(i) =& 1 + \mathbf{1}_{\{i=0\}} + \int_0^t \mathbf{1}_{\{\check{\zeta}_{s-}^{\lambda,\pi}(i)=0\}}\,\mathrm{d}N_s^S(i)\\ &+ \int_0^t \mathbf{1}_{\{\check{\zeta}_{s-}^{\lambda,\pi}(i+1)=2,\check{\zeta}_{s-}^{\lambda,\pi}(i)=1\}}\,\mathrm{d}N_s^P(i+1) + \int_0^t \mathbf{1}_{\{\check{\zeta}_{s-}^{\lambda,\pi}(i-1)=2,\check{\zeta}_{s-}^{\lambda,\pi}(i)=1\}}\,\mathrm{d}N_s^P(i-1)\\ &- 2\int_0^t \mathbf{1}_{\{\check{\zeta}_{s-}^{\lambda,\pi}(i)=2\}}\,\mathrm{d}N_s^P(i).\end{aligned}$$

Roughly, the process $(\check{\zeta}_t^{\lambda,\pi}(i))_{t\geq 0, i\in\mathbb{Z}}$ starts from an occupied initial configuration and a match falls on the site 0 at time 0. Afterwards the fire spreads into $\mathbb{Z}$. We are interested in the space-time position of burning trees (i.e. $(i,t)\in\mathbb{Z}\times[0,\infty)$ such that $\check{\zeta}_t^{\lambda,\pi}(i)=2$), when $\lambda\to 0$ and $\pi\to\infty$ in the different regimes.

We set, for $t\geq 0$,

$$i_t^+ = \max\left\{i\geq 0 : \check{\zeta}_t^{\lambda,\pi}(i)=2\right\} \tag{II.4.1}$$

$$i_t^- = \min\left\{i\leq 0 : \check{\zeta}_t^{\lambda,\pi}(i)=2\right\} \tag{II.4.2}$$

the right and the left fronts at time t. Observe that $(i_t^+)_{t\geq 0}$ and $(-i_t^-)_{t\geq 0}$ are two Poisson processes with intensity π. For $i\in\mathbb{Z}$, we set

$$\begin{aligned}T_i &= \inf\left\{s\geq 0 : \check{\zeta}_s^{\lambda,\pi}(i)=2\right\} \qquad\qquad \text{(II.4.3)}\\ &= \begin{cases}\inf\{s\geq 0 : i_s^+ = i\} & \text{if } i\geq 0,\\ \inf\{s\geq 0 : i_s^- = i\} & \text{if } i\leq 0,\end{cases}\end{aligned}$$

which represents the first time that the site $i\in\mathbb{N}$ is burning. We clearly have for all $t\geq 0$,

$$\check{\zeta}_t^{\lambda,\pi}(i_t^-) = 2 = \check{\zeta}_t^{\lambda,\pi}(i_t^+)$$

and for all $i\notin [\![i_t^-, i_t^+]\!]$,

$$\check{\zeta}_t^{\lambda,\pi}(i) = 1.$$

In this section, we will show that burning trees at some time t are *concentrated* around i_t^+ and i_t^-. We say that a site i is a *spark* at time t if it is a burning tree such that $i\notin\{i_t^-, i_t^+\}$.

We recall that $\mathbf{a}_\lambda = \log(1/\lambda)$, $\mathbf{n}_\lambda = \lfloor 1/(\lambda \mathbf{a}_\lambda) \rfloor$ and we introduce

$$\varepsilon_\lambda = \frac{1}{\mathbf{a}_\lambda^3}. \tag{II.4.4}$$

For $B > 0$, we finally set $B_\lambda = \lfloor B\mathbf{n}_\lambda \rfloor$.

The following Definition will be usefull.

Definition II.4.1. *Let $p \geq 0$. In the rest of the paper, we will say that* a statement $\mathcal{S}(\lambda, \pi)$ holds for all (λ, π) sufficiently close to the regime $\mathcal{R}(p)$ *if there are $\varepsilon_0 > 0$ and $\lambda_0 \in (0\,,1)$ such that for all $\lambda \in (0\,,\lambda_0)$ and all $\pi \geq 1$ such that $\left|\frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} - p\right| < \varepsilon_0$, the statement $\mathcal{S}(\lambda,\pi)$ holds.*

Similarly, let $z_0 \in [0\,,1]$. We will say that a statement $\mathcal{S}(\lambda, \pi)$ holds for all (λ, π) sufficiently close to the regime $\mathcal{R}(\infty, z_0)$ *if there are $\varepsilon_0 > 0$, $\lambda_0 \in (0\,,1)$ and $K_0 > 0$ such that for all $\lambda \in (0\,,\lambda_0)$ and all $\pi \geq 1$ such that $\frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} > K_0$ and $\left|\frac{\log(\pi)}{\log(1/\lambda)} - z_0\right| < \varepsilon_0$, the statement $\mathcal{S}(\lambda,\pi)$ holds.*

II.4.1. Propagation lemma in the intermediate regime

We first study the propagation in the regime $\mathcal{R}(p)$, for some $p > 0$.

Lemma II.4.2. *Let $p > 0, T > 0$. There exists an event $\Omega_{\lambda,\pi}^{P,T}$ depending only on the Poisson processes $(N_t^S(i), N_t^P(i))_{t\in[0,\mathbf{a}_\lambda(T+\varepsilon_\lambda)], i\in[\![-\lfloor \mathbf{a}_\lambda\pi(T+\varepsilon_\lambda)\rfloor, \lfloor \mathbf{a}_\lambda\pi(T+\varepsilon_\lambda)\rfloor]\!]}$ such that*

$$\begin{aligned}\Omega_{\lambda,\pi}^{P,T} \subset \{&\textit{At any time } t \in [0\,,\mathbf{a}_\lambda T], \textit{any burning tree belongs to}\\ &[\![-\lfloor (t+\mathbf{a}_\lambda\varepsilon_\lambda)\pi\rfloor\,, -\lfloor (t-\mathbf{a}_\lambda\varepsilon_\lambda)\pi\rfloor]\!] \cup [\![\lfloor (t-\mathbf{a}_\lambda\varepsilon_\lambda)\pi\rfloor\,,\lfloor (t+\mathbf{a}_\lambda\varepsilon_\lambda)\pi\rfloor]\!]\\ &\qquad\qquad \textit{and is either } i_t^+ \textit{ or } i_t^- \textit{ or has vacant neighbors}\},\end{aligned}$$

where the event on the right concerns $(\check{\zeta}_t^{\lambda,\pi}(i))_{t\geq 0, i\in\mathbb{Z}}$, and

$$\lim_{\lambda,\pi} \mathbb{P}\left[\Omega_{\lambda,\pi}^{P,T}\right] = 1$$

when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$.

Proof. Recall that a *spark* at time t is a burning tree i such that $i \notin \{i_t^-\,, i_t^+\}$. We say that a site i *propagates for the first time* when the first fire at i extinguishes and spreads to its neighbors (if they are occupied). Observe that for $i \geq 0$, this happens at time T_{i+1}, while for $i \leq 0$, this happens at time T_{i-1}.

Consider, for $i \geq 0$, the events

$$\begin{aligned}\Omega_i^1 = \{&i \text{ remains vacant from the instant at which it propagates for the first time}\\ &\text{until the instant at which the fire in } i+1 \text{ propagates for the first time}\}\end{aligned} \tag{II.4.5}$$

and

$$\begin{aligned}\Omega_i^2 = \{&i \text{ is occupied when the fire in } i+1 \text{ propagates for the first time,}\\ &\text{but then, } i \text{ burns for the second time during less than } \mathbf{a}_\lambda\varepsilon_\lambda/4\\ &\text{and no seed has fallen on its neighbors } i-1,\, i+1\\ &\text{from the instant they burnt for the first time until } i \text{ propagates for the second time}\}\end{aligned} \tag{II.4.6}$$

and similar events for $i \leq 0$ (replace $i+1$ by $i-1$). Recall (II.4.1), (II.4.2) and remark that the event on the right hand side in Lemma II.4.2 contains the event

$$\begin{aligned}\Omega_{\lambda,\pi}^{P,T} = \left\{\sup_{t\in[0,\mathbf{a}_\lambda T]} \left|i_t^+ - \pi t\right| \leq \frac{\mathbf{a}_\lambda\pi\varepsilon_\lambda}{2}\right\} \cap \left\{\sup_{t\in[0,\mathbf{a}_\lambda T]} \left|i_t^- + \pi t\right| \leq \frac{\mathbf{a}_\lambda\pi\varepsilon_\lambda}{2}\right\}\\ \cap \{\forall i \in [\![i_{\mathbf{a}_\lambda T}^- + 1\,, i_{\mathbf{a}_\lambda T}^+ - 1]\!], \Omega_i^1 \text{ or } \Omega_i^2 \text{ is realized}\}.\end{aligned}$$

Indeed, the two first terms ensure that the right and the left fronts at time $t \in [0\,, \mathbf{a}_\lambda T]$ belongs respectively to

$$[\![\lfloor(t - \mathbf{a}_\lambda\varepsilon_\lambda/2)\pi\rfloor\,, \lfloor(t + \mathbf{a}_\lambda\varepsilon_\lambda/2)\pi\rfloor]\!]$$

and

$$[\![-\lfloor(t + \mathbf{a}_\lambda\varepsilon_\lambda/2)\pi\rfloor\,, -\lfloor(t - \mathbf{a}_\lambda\varepsilon_\lambda/2)\pi\rfloor]\!].$$

This in particular implies that for all $i \in [\![-\lfloor(T - \varepsilon_\lambda/2)\mathbf{a}_\lambda\pi\rfloor\,, \lfloor(T - \varepsilon_\lambda/2)\mathbf{a}_\lambda\pi\rfloor]\!]$,

$$T_i \in \left[\frac{|i|}{\pi} - \frac{\mathbf{a}_\lambda\varepsilon_\lambda}{2}, \frac{|i|}{\pi} + \frac{\mathbf{a}_\lambda\varepsilon_\lambda}{2}\right].$$

The last term says that either i remains vacant until $i+1$ propagates (i.e. there is no spark) or a seed has fallen on i but then i has vacant neighbors when it propagates for the second time (i.e. the spark has a size 1). Finally remark that on $\Omega_{\lambda,\pi}^{P,T}$, for $t \in [0\,, \mathbf{a}_\lambda T]$,

$$\left\{0 \leq i \leq i_t^+ : T_{i+2} + \frac{\mathbf{a}_\lambda\varepsilon_\lambda}{4} \geq t\right\} \subset [\![\lfloor(t - \mathbf{a}_\lambda\varepsilon_\lambda)\pi\rfloor\,, i_t^+]\!]$$

and

$$\left\{0 \geq i \geq i_t^- : T_{i-2} + \frac{\mathbf{a}_\lambda\varepsilon_\lambda}{4} \geq t\right\} \subset [\![i_t^-\,, -\lfloor(t - \mathbf{a}_\lambda\varepsilon_\lambda)\pi\rfloor]\!],$$

thus a burning tree (i.e. a front or a spark) necessarily belongs to

$$[\![\lfloor(t - \mathbf{a}_\lambda\varepsilon_\lambda)\pi\rfloor\,, \lfloor(t + \mathbf{a}_\lambda\varepsilon_\lambda)\pi\rfloor]\!] \cup [\![-\lfloor(t + \mathbf{a}_\lambda\varepsilon_\lambda)\pi\rfloor\,, -\lfloor(t - \mathbf{a}_\lambda\varepsilon_\lambda)\pi\rfloor]\!],$$

as desired.

Clearly, $\Omega_{\lambda,\pi}^{P,T}$ depends only on the Poisson processes $(N_t^S(i), N_t^P(i))_{t\geq 0, i\in\mathbb{Z}}$ through $t \in [0\,, \mathbf{a}_\lambda(T + \varepsilon_\lambda)]$ and $i \in [\![-\lfloor\mathbf{a}_\lambda\pi(T + \varepsilon_\lambda)\rfloor\,, \lfloor\mathbf{a}_\lambda\pi(T + \varepsilon_\lambda)\rfloor]\!]$. It remains to prove that $\mathbb{P}\left[\Omega_{\lambda,\pi}^{P,T}\right]$ tends to 1 when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$.

Since $(i_t^+)_{t\geq 0}$ and $(-i_t^-)_{t\geq 0}$ are two Poisson processes with intensity π, the maximal inequality for martingales gives

$$\begin{aligned}\mathbb{P}\left[\sup_{t\in[0,\mathbf{a}_\lambda T]}\left|i_t^- + \pi t\right| > \frac{\mathbf{a}_\lambda \pi \varepsilon_\lambda}{2}\right] &= \mathbb{P}\left[\sup_{t\in[0,\mathbf{a}_\lambda T]}\left|i_t^+ - \pi t\right| > \frac{\mathbf{a}_\lambda \pi \varepsilon_\lambda}{2}\right] \\ &\leq \left(\frac{2}{\mathbf{a}_\lambda \pi \varepsilon_\lambda}\right)^4 \times (3(\mathbf{a}_\lambda \pi T)^2 + \mathbf{a}_\lambda \pi T) \\ &\leq \frac{16T^2}{(\mathbf{a}_\lambda \pi \varepsilon_\lambda^2)^2} = \frac{16T^2\mathbf{a}_\lambda^{10}}{\pi^2} \end{aligned} \tag{II.4.7}$$

which tends to 0 when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$.

Next, for all $i \in \mathbb{Z}$, we have

$$\mathbb{P}\left[\Omega_i^1\right] = \frac{\pi}{1+\pi} \tag{II.4.8}$$

because seeds fall on i at rate 1 while the fire on $i+1$ propagates at rate π.

Now, for all $i \geq 0$, we set

$$\begin{aligned} X_i &= \inf\left\{s > T_{i+1} : N_s^S(i) - N_{T_{i+1}}^S(i) > 0\right\} - T_{i+1}, \\ Y_i^1 &= T_{i+1} - T_i, \\ Y_i^2 &= \inf\left\{s > T_{i+2} : N_s^P(i) - N_{T_{i+2}}^P(i) > 0\right\} - T_{i+2}. \end{aligned}$$

Let $i \geq 0$. By construction, at time T_i, the site i is burning and propagates to neighbors at time T_{i+1}. Thus, X_i is the time we have to wait for a seed to fall again on i after it propagates for the first time. Furthermore, Y_i^1 stands for the duration that i is burning for the first time. If a seed falls on i before T_{i+2}, that is before the burning tree $i+1$ propagates, then i becomes again burning at time T_{i+2} and burns during $[T_{i+2}\,, T_{i+2} + Y_i^2)$.

The random variables $(X_i)_{i\in\mathbb{N}}$ are exponential random variables with parameter 1 and the random variables $(Y_i^1)_{i\in\mathbb{N}}$ and $(Y_i^2)_{i\in\mathbb{N}}$ are exponential random variables with parameter π. All these random variables are independent.

Then observe that, for all $i \geq 0$

$$\Omega_i^2 = \left(\{X_i \leq Y_{i+1}^1\} \cap \{Y_i^2 < \frac{\mathbf{a}_\lambda \varepsilon_\lambda}{4}\} \cap \{X_{i-1} > Y_i^1 + Y_{i+1}^1 + Y_i^2\} \cap \{X_{i+1} > Y_i^2\}\right). \tag{II.4.9}$$

We have by independence

$$\begin{aligned}\mathbb{P}\left[\Omega_i^2 \,\middle|\, Y_i^1, Y_{i+1}^1, Y_i^2\right] &= (1 - e^{-Y_{i+1}^1}) \times \mathbf{1}_{\{Y_i^2 \leq \frac{\mathbf{a}_\lambda \varepsilon_\lambda}{4}\}} \times e^{-(Y_i^1 + Y_{i+1}^1 + Y_i^2)} \times e^{-Y_i^2} \\ &= (1 - e^{-Y_{i+1}^1}) \times e^{-Y_{i+1}^1} \times e^{-Y_i^1} \times e^{-2Y_i^2} \times \mathbf{1}_{\{Y_i^2 \leq \frac{\mathbf{a}_\lambda \varepsilon_\lambda}{4}\}}. \end{aligned}$$

Integrating,

$$\begin{aligned}\mathbb{P}\left[\Omega_i^2\right] &= \pi^3 \int_0^\infty (1 - e^{-x}) e^{-(\pi+1)x}\,\mathrm{d}x \times \int_0^\infty e^{-(\pi+1)y}\,\mathrm{d}y \times \int_0^{\mathbf{a}_\lambda \varepsilon_\lambda/4} e^{-(\pi+2)z}\,\mathrm{d}z \\ &= \frac{\pi^3}{(1+\pi)^2(2+\pi)^2}(1 - e^{-(2+\pi)\mathbf{a}_\lambda \varepsilon_\lambda/4}). \end{aligned} \tag{II.4.10}$$

Finally, note that, in the regime $\mathcal{R}(p)$,

$$\begin{aligned}\mathbb{P}\left[\Omega_i^1 \cup \Omega_i^2\right] = \mathbb{P}\left[\Omega_i^1\right] + \mathbb{P}\left[\Omega_i^2\right] &= \frac{\pi}{1+\pi} + \frac{\pi^3}{(1+\pi)^2(2+\pi)^2}(1 - e^{-(2+\pi)\mathbf{a}_\lambda\varepsilon_\lambda/4}) \\ &= 1 - \frac{5\pi^2 + 8\pi + 4 + \pi^3 e^{-(2+\pi)\mathbf{a}_\lambda\varepsilon_\lambda/4}}{(1+\pi)^2(2+\pi)^2} \\ &\geq 1 - \frac{\alpha}{\pi^2}\end{aligned}$$

for some constant $\alpha > 0$, because $e^{-(2+\pi)\mathbf{a}_\lambda\varepsilon_\lambda/4} \ll 1/\pi$ when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$ (indeed, $\pi \sim 1/(p\lambda\log^2(1/\lambda))$ whence $(2+\pi)\mathbf{a}_\lambda\varepsilon_\lambda \simeq 1/(p\lambda\log^3(1/\lambda)))$.

Similar computations hold for $i \leq 0$.

Consequently, the probability of $\{\forall i \in [\![i^-_{\mathbf{a}_\lambda T}+1\,, i^+_{\mathbf{a}_\lambda T}-1]\!], \Omega_i^1$ or Ω_i^2 is realized$\}$ knowing $\left\{\sup_{t\in[0,\mathbf{a}_\lambda T]}\left|i_t^+ - \pi t\right| \leq \frac{\mathbf{a}_\lambda\pi\varepsilon_\lambda}{2}\right\} \cap \left\{\sup_{t\in[0,\mathbf{a}_\lambda T]}\left|i_t^- + \pi t\right| \leq \frac{\mathbf{a}_\lambda\pi\varepsilon_\lambda}{2}\right\}$ is bounded from below by

$$\begin{aligned}1 - \sum_{i=-\lfloor\mathbf{a}_\lambda\pi(T+\varepsilon_\lambda)\rfloor}^{\lfloor\mathbf{a}_\lambda\pi(T+\varepsilon_\lambda)\rfloor} \mathbb{P}\left[(\Omega_i^1 \cup \Omega_i^2)^c\right] &= 1 - \sum_{i=-\lfloor\mathbf{a}_\lambda\pi(T+\varepsilon_\lambda)\rfloor}^{\lfloor\mathbf{a}_\lambda\pi(T+\varepsilon_\lambda)\rfloor} (1 - \mathbb{P}\left[\Omega_i^1\right] - \mathbb{P}\left[\Omega_i^2\right]) \\ &\geq 1 - \alpha\frac{\mathbf{a}_\lambda\pi(T+1)}{\pi^2} = 1 - \alpha_T\frac{\mathbf{a}_\lambda}{\pi} \qquad \text{(II.4.11)}\end{aligned}$$

which tends to 1 when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$. Gathering (II.4.7) and (II.4.11) concludes the proof of Lemma II.4.2. ☐

II.4.2. Propagation lemma in the regime $\mathcal{R}(0)$

For all $A > 0$, we set

$$\varkappa_{\lambda,\pi}^A = \frac{\mathbf{n}_\lambda A}{\mathbf{a}_\lambda\pi} + \varepsilon_\lambda \qquad \text{(II.4.12)}$$

which tends to 0 when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(0)$.

Lemma II.4.3. *Let $A, B > 0$. There exists an event $\Omega_{\lambda,\pi}^{P,A,B}$ depending only on the Poisson processes $(N_t^S(i), N_t^P(i))_{t\in[0,\mathbf{a}_\lambda\varkappa_{\lambda,\pi}^{A\vee B}], i\in[\![-A_\lambda-\mathbf{m}_\lambda, B_\lambda+\mathbf{m}_\lambda]\!]}$ such that*

$$\begin{aligned}\Omega_{\lambda,\pi}^{P,A,B} \subset \{&\textit{There is no more burning tree in } [\![-A_\lambda\,, B_\lambda]\!] \textit{ at time } \mathbf{a}_\lambda\varkappa_{\lambda,\pi}^{A\vee B} \\ &\textit{and a burning tree in } [\![-A_\lambda\,, B_\lambda]\!] \textit{ at some time } 0 \leq t \leq \mathbf{a}_\lambda\varkappa_{\lambda,\pi}^{A\vee B} \\ &\qquad\textit{is either } i_t^+ \textit{ or } i_t^- \textit{ or has vacant neighbors}\}\end{aligned}$$

where the event on the right concerns $(\check{\zeta}_t^{\lambda,\pi}(i))_{t\geq 0, i\in\mathbb{Z}}$, and

$$\lim_{\lambda,\pi} \mathbb{P}\left[\Omega_{\lambda,\pi}^{P,A,B}\right] = 1$$

when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(0)$.

Proof. Recall (II.4.3), (II.4.5) and (II.4.6). We set

$$\begin{aligned}\Omega_{\lambda,\pi}^{P,A,B} = \left\{T_{B_\lambda+\mathbf{m}_\lambda} \le \frac{\mathbf{n}_\lambda B}{\pi} + \frac{\mathbf{a}_\lambda\varepsilon_\lambda}{2}\right\} \cap \left\{T_{-A_\lambda-\mathbf{m}_\lambda} \le \frac{\mathbf{n}_\lambda A}{\pi} + \frac{\mathbf{a}_\lambda\varepsilon_\lambda}{2}\right\} \\ \cap \bigcap_{i\in[\![-A_\lambda-\mathbf{m}_\lambda+1,B_\lambda+\mathbf{m}_\lambda-1]\!]} (\Omega_i^1 \cup \Omega_i^2) \\ \cap \left\{\exists i \in [\![-A_\lambda - \mathbf{m}_\lambda + 1\,, -A_\lambda]\!], N^S_{\mathbf{a}_\lambda\varkappa_{\lambda,\pi}^{A\vee B}}(i) = 0\right\} \\ \cap \left\{\exists i \in [\![B_\lambda\,, B_\lambda + \mathbf{m}_\lambda - 1]\!], N^S_{\mathbf{a}_\lambda\varkappa_{\lambda,\pi}^{A\vee B}}(i) = 0\right\}.\end{aligned}$$

Observe now that the event on the right hand side in Lemma II.4.3 contains the event $\Omega_{\lambda,\pi}^{P,A,B}$.

Indeed, the two first terms ensure that the left and the right fronts have left the zone $[\![-A_\lambda\,, B_\lambda]\!]$ at time $\mathbf{a}_\lambda\varkappa_{\lambda,\pi}^{A\vee B}$ whereas the third term ensures, as in Lemma II.4.2, that a spark burns not for a long time and has vacants neighbors. The two last terms prevent from a return of a fire until $\mathbf{a}_\lambda\varkappa_{\lambda,\pi}^{A\vee B}$.

It remains to prove that $\mathbb{P}\left[\Omega_{\lambda,\pi}^{P,A,B}\right]$ tends to 1 as $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(0)$. First, observe that $T_{B_\lambda+\mathbf{m}_\lambda}$ is a sum of $B_\lambda+\mathbf{m}_\lambda$ i.i.d. exponential random variables with parameter π. Then, Chebyshev's inequality implies

$$\begin{aligned}\mathbb{P}\left[T_{B_\lambda+\mathbf{m}_\lambda} > \frac{\mathbf{n}_\lambda B}{\pi} + \frac{\mathbf{a}_\lambda\varepsilon_\lambda}{2}\right] \le \mathbb{P}\left[\left|T_{B_\lambda+\mathbf{m}_\lambda} - \frac{\mathbf{n}_\lambda B}{\pi}\right| > \frac{\mathbf{a}_\lambda\varepsilon_\lambda}{2}\right] \le \frac{4}{(\mathbf{a}_\lambda\varepsilon_\lambda)^2}\frac{B_\lambda+\mathbf{m}_\lambda}{\pi^2} \\ \le C_B\frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda\pi}\frac{1}{\mathbf{a}_\lambda\pi\varepsilon_\lambda^2}\end{aligned}$$

which tends to 0 when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(0)$. Similar computation of course holds for $T_{-A_\lambda-\mathbf{m}_\lambda}$.

A basic calculation, as in (II.4.11), shows that (because it also holds true that $e^{-(2+\pi)\mathbf{a}_\lambda\varepsilon_\lambda/4} \ll 1/\pi$ in the regime $\mathcal{R}(0)$)

$$\mathbb{P}\left[\bigcap_{i\in[\![-A_\lambda-\mathbf{m}_\lambda+1,B_\lambda+\mathbf{m}_\lambda-1]\!]} (\Omega_i^1 \cup \Omega_i^2)\right] \ge 1 - \alpha\frac{\mathbf{a}_\lambda}{\pi} \quad \text{(for some } \alpha = \alpha(A,B) > 0\text{)},$$

which tends to 1 when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(0)$.

Finally, as soon as $\varkappa_{\lambda,\pi}^{A\vee B} \le \frac{1}{2}$, it holds that, using space stationarity,

$$\begin{aligned}\mathbb{P}\left[\exists i \in [\![B_\lambda\,, B_\lambda+\mathbf{m}_\lambda-1]\!], N^S_{\mathbf{a}_\lambda\varkappa_{\lambda,\pi}^{A\vee B}}(i) = 0\right] \ge \mathbb{P}\left[\exists i \in [\![0\,, \mathbf{m}_\lambda - 1]\!], N^S_{\mathbf{a}_\lambda/2}(i) = 0\right] \\ = 1 - (1 - e^{-\mathbf{a}_\lambda/2})^{\mathbf{m}_\lambda-1} \simeq 1 - e^{-\sqrt{\lambda}(\mathbf{m}_\lambda-1)}\end{aligned}$$

which tends to 1 when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(0)$. $\square$

II.4.3. Propagation lemma in the regime $\mathcal{R}(\infty, z_0)$

Let $z_0 \in [0\,,1]$. We first introduce, for $\lambda \in (0\,,1]$ and $\gamma \in (0\,,1)$,

$$\mathbf{m}_\lambda^\gamma = \left\lfloor \frac{\gamma}{\lambda^{\gamma+(1-\gamma)z_0}\mathbf{a}_\lambda} \right\rfloor .$$

For $z_0 = 1$, $\mathbf{m}_\lambda^\gamma$ is nothing but $\lfloor \gamma \mathbf{n}_\lambda \rfloor$. For $z_0 \in [0\,,1)$ and $\gamma \in (0\,,1)$, observe that

$$z_0 < \gamma + (1-\gamma)z_0 < 1,$$

so that $\mathbf{m}_\lambda^\gamma \ll \mathbf{n}_\lambda$. In any cases, we have $\mathbf{m}_\lambda^\gamma / \mathbf{n}_\lambda \leq \gamma$.

Lemma II.4.4. *Let $T > 0$, $z_0 \in [0\,,1]$ and $\gamma \in (0\,,1)$. There exists an event $\Omega_{\lambda,\pi}^{P,T,\gamma}$ depending only on the Poisson processes $(N_t^S(i), N_t^P(i))_{t \in [0,\mathbf{a}_\lambda T], i \in [\![-\mathbf{m}_\lambda^\gamma, \mathbf{m}_\lambda^\gamma]\!]}$, such that*

$$\Omega_{\lambda,\pi}^{P,T,\gamma} \subset \{ i_{\mathbf{a}_\lambda T}^+ \text{ and } i_{\mathbf{a}_\lambda T}^- \text{ belong to } [\![-\mathbf{m}_\lambda^\gamma\,, \mathbf{m}_\lambda^\gamma]\!] \},$$

where the event on the right concerns the process $(\check{\zeta}_t^{\lambda,\pi}(i))_{t \geq 0, i \in \mathbb{Z}}$, and

$$\lim_{\lambda,\pi} \mathbb{P}\left[\Omega_{\lambda,\pi}^{P,T,\gamma}\right] = 1$$

when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(\infty, z_0)$.

Proof. Recall (II.4.1) and (II.4.2). We define

$$\Omega_{\lambda,\pi}^{P,T,\gamma} = \{0 \leq i_{\mathbf{a}_\lambda T}^+ \leq \mathbf{m}_\lambda^\gamma\} \cap \{-\mathbf{m}_\lambda^\gamma \leq i_{\mathbf{a}_\lambda T}^- \leq 0\},$$

which clearly implies that $i_{\mathbf{a}_\lambda T}^+$ and $i_{\mathbf{a}_\lambda T}^-$ belong to $[\![-\mathbf{m}_\lambda^\gamma\,, \mathbf{m}_\lambda^\gamma]\!]$. Since $i_{\mathbf{a}_\lambda T}^+$ and $-i_{\mathbf{a}_\lambda T}^-$ are two Poisson random variables with parameter $\mathbf{a}_\lambda \pi T$, Markov's inequality shows that

$$\mathbb{P}\left[i_{\mathbf{a}_\lambda T}^- < -\mathbf{m}_\lambda^\gamma\right] = \mathbb{P}\left[i_{\mathbf{a}_\lambda T}^+ > \mathbf{m}_\lambda^\gamma\right] \leq \frac{\mathbf{a}_\lambda \pi T}{\mathbf{m}_\lambda^\gamma} \simeq \frac{T}{\gamma} \mathbf{a}_\lambda^2 \pi \lambda^{\gamma+(1-\gamma)z_0},$$

which tends to 0 when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(\infty, z_0)$. Indeed, for $z_0 = 1$, then $\frac{T}{\gamma}\mathbf{a}_\lambda^2 \pi \lambda = \frac{T}{\gamma}\frac{\mathbf{a}_\lambda \pi}{\mathbf{n}_\lambda}$ tends to 0 (it is the definition of the regime $\mathcal{R}(\infty, 1)$), while, for $z_0 \in [0\,,1)$, since $z_0 < \gamma + (1-\gamma)z_0 < 1$, then $\frac{T}{\gamma}\mathbf{a}_\lambda^2 \pi \lambda^{\gamma+(1-\gamma)z_0} = \frac{T}{\gamma}\frac{\mathbf{a}_\lambda^2 \pi}{\lambda^{z_0}} \lambda^{(1-z_0)\gamma}$ tends to 0, because $\log(\pi)/\log(1/\lambda)$ tends to z_0. $\square$

For $z \in (0\,,1)$, we next define

$$\kappa_{\lambda,\pi}^z = \frac{1}{\lambda^z \mathbf{a}_\lambda \pi} + \varepsilon_\lambda. \tag{II.4.13}$$

Observe that, if $0 < z < z_0$, then $\mathbf{a}_\lambda \kappa_{\lambda,\pi}^z$ tends to 0 when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(\infty, z_0)$.

Lemma II.4.5. *For all $z_0 \in (0\,,1]$ and all $z \in (0\,,z_0)$, there exists an event $\Omega^{P,z}_{\lambda,\pi}$, depending only on the Poisson processes $(N^S_t(i), N^P_t(i))_{t\in[0,\mathbf{a}_\lambda T], i\in[\![-\mathbf{m}^\gamma_\lambda,\mathbf{m}^\gamma_\lambda]\!]}$, such that*

$$\Omega^{P,z}_{\lambda,\pi} \subset \{i^+_{\mathbf{a}_\lambda\kappa^z_{\lambda,\pi}} \textit{ and } -i^-_{\mathbf{a}_\lambda\kappa^z_{\lambda,\pi}} \textit{are greater than } \lfloor\lambda^{-z}\rfloor$$
$$\textit{and all } i \in [\![i^-_{\mathbf{a}_\lambda\kappa^z_{\lambda,\pi}}+1\,, i^+_{\mathbf{a}_\lambda\kappa^z_{\lambda,\pi}}-1]\!] \textit{ burns exactly once before } \mathbf{a}_\lambda\kappa^z_{\lambda,\pi}\},$$

where the event on the right concerns the process $(\check{\zeta}^{\lambda,\pi}_t(i))_{t\geq 0, i\in\mathbb{Z}}$, and

$$\lim_{\lambda,\pi} \mathbb{P}\left[\Omega^{P,z}_{\lambda,\pi}\right] = 1$$

when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(\infty, z_0)$.

Proof. Let $z \in (0\,,z_0)$. Recall (II.4.1), (II.4.2), (II.4.5) and remark that

$$\mathbf{a}_\lambda\pi\kappa^z_{\lambda,\pi} = \lambda^{-z} + \mathbf{a}_\lambda\pi\varepsilon_\lambda.$$

We define

$$\Omega^{P,z}_{\lambda,\pi} = \left\{i^+_{\mathbf{a}_\lambda\kappa^z_{\lambda,\pi}} \in [\![\lfloor\lambda^{-z}\rfloor\,, \lfloor\lambda^{-z}+2\mathbf{a}_\lambda\pi\varepsilon_\lambda\rfloor]\!]\right\} \cap \left\{i^-_{\mathbf{a}_\lambda\kappa^z_{\lambda,\pi}} \in [\![-\lfloor\lambda^{-z}-2\mathbf{a}_\lambda\pi\varepsilon_\lambda\rfloor\,, \lfloor\lambda^{-z}\rfloor]\!]\right\}$$
$$\cap \bigcap_{i\in[\![-\lfloor\lambda^{-z}+2\mathbf{a}_\lambda\pi\varepsilon_\lambda\rfloor, \lfloor\lambda^{-z}+2\mathbf{a}_\lambda\pi\varepsilon_\lambda\rfloor]\!]} \Omega^1_i.$$

Observe that the event on the right hand side in Lemma II.4.5 contains the event $\Omega^{P,z}_{\lambda,\pi}$. Indeed, as in the proof of Lemma II.4.2, the two first terms situate the left and the right fronts. The third term ensures that there is no spark in the zone

$$[\![-\lfloor\mathbf{a}_\lambda\pi(\kappa^z_{\lambda,\pi}+\varepsilon_\lambda)\rfloor\,, \lfloor\mathbf{a}_\lambda\pi(\kappa^z_{\lambda,\pi}+\varepsilon_\lambda)\rfloor]\!] \supset [\![i^-_{\mathbf{a}_\lambda\kappa^z_{\lambda,\pi}}\,, i^+_{\mathbf{a}_\lambda\kappa^z_{\lambda,\pi}}]\!] \supset [\![-\lfloor\lambda^{-z}\rfloor\,, \lfloor\lambda^{-z}\rfloor]\!].$$

Since $i^+_{\mathbf{a}_\lambda\kappa^z_{\lambda,\pi}}$ and $-i^-_{\mathbf{a}_\lambda\kappa^z_{\lambda,\pi}}$ are two Poisson random variables with parameter $\mathbf{a}_\lambda\pi\kappa^z_{\lambda,\pi}$, Chebyshev's inequality shows

$$\begin{aligned}
\mathbb{P}\left[i^-_{\mathbf{a}_\lambda\kappa^z_{\lambda,\pi}} \notin [\![-\lfloor\lambda^{-z}-2\mathbf{a}_\lambda\varepsilon_\lambda\pi\rfloor\,, -\lfloor\lambda^{-z}\rfloor]\!]\right] &= \mathbb{P}\left[|i^-_{\mathbf{a}_\lambda\kappa^z_{\lambda,\pi}} + \mathbf{a}_\lambda\pi\kappa^z_{\lambda,\pi}| > \mathbf{a}_\lambda\pi\varepsilon_\lambda\right] \\
= \mathbb{P}\left[i^+_{\mathbf{a}_\lambda\kappa^z_{\lambda,\pi}} \notin [\![\lfloor\lambda^{-z}\rfloor\,, \lfloor\lambda^{-z}+2\mathbf{a}_\lambda\varepsilon_\lambda\pi\rfloor]\!]\right] &= \mathbb{P}\left[\left|i^+_{\mathbf{a}_\lambda\kappa^z_{\lambda,\pi}} - \mathbf{a}_\lambda\pi\kappa^z_{\lambda,\pi}\right| > \mathbf{a}_\lambda\pi\varepsilon_\lambda\right] \\
&\leq \frac{\mathbf{a}_\lambda\pi\kappa^z_{\lambda,\pi}}{(\mathbf{a}_\lambda\varepsilon_\lambda\pi)^2} = \frac{\kappa^z_{\lambda,\pi}}{\mathbf{a}_\lambda\pi\varepsilon^2_\lambda} = \kappa^z_{\lambda,\pi}\frac{\mathbf{a}^3_\lambda}{\pi}
\end{aligned}$$

which again tends to 0 when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(\infty, z_0)$ (because $\log(\pi) \sim z_0\mathbf{a}_\lambda$).

Finally, we still have $\mathbb{P}\left[\Omega^1_i\right] = \frac{\pi}{1+\pi}$, recall (II.4.8), whence

$$\mathbb{P}\left[\bigcap_{i\in[\![-\lfloor\mathbf{a}_\lambda\pi(\kappa^z_{\lambda,\pi}+\varepsilon_\lambda)\rfloor, \lfloor\mathbf{a}_\lambda\pi(\kappa^z_{\lambda,\pi}+\varepsilon_\lambda)\rfloor]\!]} \Omega^1_i\right] = \left(\frac{\pi}{1+\pi}\right)^{2\lfloor\mathbf{a}_\lambda\pi(\kappa^z_{\lambda,\pi}+\varepsilon_\lambda)\rfloor+1} \simeq e^{-2\mathbf{a}_\lambda(\kappa^z_{\lambda,\pi}+\varepsilon_\lambda)}$$

which tends to 1 when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(\infty, z_0)$. This concludes the proof of Lemma II.4.5. $\square$

II.4.4. Application to the $(\lambda,\pi)-$FFP

We next give some useful definitions.

Definition II.4.6. *Consider two families of Poisson processes* $(N_t^S(i))_{t\geq 0, i\in\mathbb{Z}}$ *and* $(N_t^P(i))_{t\geq 0, i\in\mathbb{Z}}$ *with respective rates* 1 *and* π*, all these processes being independent. Let* $(x_0,t_0)\in\mathbb{R}\times\mathbb{R}_+$*. We call*

- propagation process ignited at (x_0,t_0) *the process* $(\check{\zeta}_t^{\lambda,\pi,0}(i))_{t\geq 0, i\in\mathbb{Z}}$ *built using the seed processes family* $(N_t^{S,0}(i))_{t\geq 0, i\in\mathbb{Z}} = (N^S_{t+\mathbf{a}_\lambda t_0}(i+\lfloor \mathbf{n}_\lambda x_0\rfloor)-N^S_{\mathbf{a}_\lambda t_0}(i+\lfloor \mathbf{n}_\lambda x_0\rfloor))_{t\geq 0, i\in\mathbb{Z}}$ *and the propagation processes family* $(N_t^{P,0}(i))_{t\geq 0, i\in\mathbb{Z}} = (N^P_{t+\mathbf{a}_\lambda t_0}(i+\lfloor \mathbf{n}_\lambda x_0\rfloor)-N^P_{\mathbf{a}_\lambda t_0}(i+\lfloor \mathbf{n}_\lambda x_0\rfloor))_{t\geq 0, i\in\mathbb{Z}}$*;*
- right and left fronts of the propagation process ignited at (x_0,t_0) *the processes* $(i_t^{0,+})_{t\geq 0}$ *and* $(i_t^{0,-})_{t\geq 0}$*, where for* $t\geq 0$

$$i_t^{0,+} = \max\left\{i\geq 0 : \check{\zeta}_t^{\lambda,\pi,0}(i)=2\right\},$$
$$i_t^{0,-} = \min\left\{i\leq 0 : \check{\zeta}_t^{\lambda,\pi,0}(i)=2\right\}.$$

  The processes $(i_t^{0,+})_{t\geq 0}$ *and* $(-i_t^{0,-})_{t\geq 0}$ *are Poisson processes with parameter* π*;*
- burning times of the propagation process ignited at (x_0,t_0) *the sequence* $(T_i^0)_{i\in\mathbb{Z}}$ *where, for* $i\in\mathbb{Z}$*,*

$$\begin{aligned} T_i^0 &= \inf\left\{s\geq 0 : \check{\zeta}_s^{\lambda,\pi,0}(i)=2\right\} \\ &= \begin{cases} \inf\left\{s\geq 0 : i_s^{0,+}=i\right\} & \text{if } i\geq 0, \\ \inf\left\{s\geq 0 : i_s^{0,-}=i\right\} & \text{if } i\leq 0. \end{cases} \end{aligned}$$

Observe that $(T_i^0)_{i\in\mathbb{Z}}$*,*$(i_t^{0,+})_{t\geq 0}$ *and* $(-i_t^{0,-})_{t\geq 0}$ *only depend on the propagation processes family* $(N_t^P(i))_{t\geq 0, i\in\mathbb{Z}}$*.*

We then reformulate Lemmas II.4.2, II.4.3, II.4.4 and II.4.5 with the previous definition.

Definition II.4.7. *Consider two families of Poisson processes* $(N_t^S(i))_{t\geq 0, i\in\mathbb{Z}}$ *and* $(N_t^P(i))_{t\geq 0, i\in\mathbb{Z}}$ *with respective rates* 1 *and* π*, all these processes being independent. Let* $(x_0,t_0)\in\mathbb{R}\times\mathbb{R}_+$ *and* $(\check{\zeta}_t^{\lambda,\pi,0}(i))_{t\geq 0, i\in\mathbb{Z}}$ *be the propagation process ignited at* (x_0,t_0)*, recall Definition II.4.6.*

- *We define, for* $T>0$*,* $\Omega^{P,T}_{\lambda,\pi}(x_0,t_0) := \Omega^{P,T}_{\lambda,\pi}$*, where* $\Omega^{P,T}_{\lambda,\pi}$ *is defined as in Lemma II.4.2, using the process* $(\check{\zeta}_t^{\lambda,\pi,0}(i))_{t\geq 0, i\in\mathbb{Z}}$*.*

  Lemma II.4.2 implies that for all $\delta>0$*,* $\mathbb{P}\left[\Omega^{P,T}_{\lambda,\pi}(x_0,t_0)\right]\geq 1-\delta$ *for all* (λ,π) *sufficiently close to the regime* $\mathcal{R}(p)$*.*

- *We define, for $A, B > 0$, $\Omega^{P,A,B}_{\lambda,\pi}(x_0, t_0) := \Omega^{P,A,B}_{\lambda,\pi}$, where $\Omega^{P,A,B}_{\lambda,\pi}$ is defined as in Lemma II.4.3, using the process $(\check{\zeta}^{\lambda,\pi,0}_t(i))_{t\geq 0, i\in\mathbb{Z}}$.*

  Lemma II.4.3 implies that for all $\delta > 0$, $\mathbb{P}\left[\Omega^{P,A,B}_{\lambda,\pi}(x_0,t_0)\right] \geq 1-\delta$ for all (λ,π) sufficiently close to the regime $\mathcal{R}(0)$.

- *We define, for $z_0 \in [0\,,1]$ and $\gamma \in (0\,,1)$, $\Omega^{P,T,\gamma}_{\lambda,\pi}(x_0,t_0) := \Omega^{P,T,\gamma}_{\lambda,\pi}$, where $\Omega^{P,T,\gamma}_{\lambda,\pi}$ is defined as in Lemma II.4.4, using the process $(\check{\zeta}^{\lambda,\pi,0}_t(i))_{t\geq 0, i\in\mathbb{Z}}$.*

  Lemma II.4.4 implies that for all $\delta > 0$, $\mathbb{P}\left[\Omega^{P,T,\gamma}_{\lambda,\pi}(x_0,t_0)\right] \geq 1-\delta$ for all (λ,π) sufficiently close to the regime $\mathcal{R}(\infty, z_0)$.

- *We define, for $z_0 \in (0\,,1]$ and $z \in (0\,, z_0)$, $\Omega^{P,z}_{\lambda,\pi}(x_0,t_0) := \Omega^{P,z}_{\lambda,\pi}$, where $\Omega^{P,z}_{\lambda,\pi}$ is defined as in Lemma II.4.5, using the process $(\check{\zeta}^{\lambda,\pi,0}_t(i))_{t\geq 0, i\in\mathbb{Z}}$.*

  Lemma II.4.5 implies that for all $\delta > 0$, $\mathbb{P}\left[\Omega^{P,z}_{\lambda,\pi}(x_0,t_0)\right] \geq 1-\delta$ for all (λ,π) sufficiently close to the regime $\mathcal{R}(\infty, z_0)$.

Finally, we define the destroyed component by a fire starting on $\lfloor \mathbf{n}_\lambda x_0 \rfloor$ at time $\mathbf{a}_\lambda t_0$. Indeed, knowing the sequence of burning times $(T_i)_{i\in\mathbb{Z}}$ and conditionally on a suitable event defined above, we can localize the set of sites which are burning by a fire.

Definition II.4.8. *Consider a family of independent Poisson processes $(N^P_t(i))_{t\geq 0, i\in\mathbb{Z}}$ with rate π. Let $(x_0, t_0) \in \mathbb{R} \times [0\,,T]$ and let $(T^0_i)_{i\in\mathbb{Z}}$ be the burning times of the propagation process ignited at (x_0,t_0). For a $\mathbb{N}-$valued process $(\eta_t(i))_{t\geq 0, i\in\mathbb{Z}}$, we define*

$$C^P((\eta_t(i))_{i\in\mathbb{Z}, t\geq 0}, (x_0,t_0)) = [\![\lfloor \mathbf{n}_\lambda x_0 \rfloor + i^g\,, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i^d]\!] \qquad \text{(II.4.14)}$$

where

$$i^g = \max\left\{ i \leq 0 : \eta_{\mathbf{a}_\lambda t_0 + T^0_i -}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i) = 0 \right\} + 1,$$
$$i^d = \min\left\{ i \geq 0 : \eta_{\mathbf{a}_\lambda t_0 + T^0_i -}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i) = 0 \right\} - 1.$$

We will use this definition with the $(\lambda,\pi)-$FFP in the following way: on a suitable event, $C^P((\eta^{\lambda,\pi}_t(i))_{i\in\mathbb{Z}, t\geq 0}, (x_0,t_0))$ is exactly the component destroyed by a match falling in $\lfloor \mathbf{n}_\lambda x_0 \rfloor$ at time $\mathbf{a}_\lambda t_0$, see the comments below.

Let now $(\eta^{\lambda,\pi}_t(i))_{t\geq 0, i\in\mathbb{Z}}$ be the $(\lambda,\pi)-$FFP. Let $(x_0,t_0) \in \mathbb{R} \times [0\,,\infty)$ be fixed in the rest of the section. Assume that a match falls in $\lfloor \mathbf{n}_\lambda x_0 \rfloor$ at some time $\mathbf{a}_\lambda t_0$. Then, on an appropriate event and regardless of the other phenomena, fires propagate with the good speed while they spread in occupied zones. Indeed, consider $(\check{\zeta}^{\lambda,\pi,0}_t(i))_{t\geq 0, i\in\mathbb{Z}}$ the propagation process ignited at (x_0,t_0), the associated right front $(i^{0,+}_t)_{t\geq 0}$ and left front $(i^{0,-}_t)_{t\geq 0}$ and the associated burning times $(T^0_i)_{i\in\mathbb{Z}}$. Remark that $T^0_{i-\lfloor \mathbf{n}_\lambda x_0 \rfloor}$ is the time needed for the fire starting in $\lfloor \mathbf{n}_\lambda x_0 \rfloor$ at time $\mathbf{a}_\lambda t_0$ to reach i.

Microscopic fire.

We describe here the effect of a microscopic fire in the discrete process in the different regimes. Let $\lambda \in (0\,,1]$ and $\pi \geq 1$.

Micro(p)**:** here we focus on the regime $\mathcal{R}(p)$, for some $p > 0$. Set

$$\kappa^0_{\lambda,\pi} = \frac{\mathbf{m}_\lambda}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda.$$

Assume that

▷ there are $-\mathbf{m}_\lambda < i_1 < 0 < i_2 < \mathbf{m}_\lambda$ such that

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1) = \eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2) = 0$$

for all $t \in [t_0\,, t_0 + \kappa^0_{\lambda,\pi}]$,

▷ there is no burning tree in $[\![\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1\,, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2]\!]$ at time $\mathbf{a}_\lambda t_0-$,

▷ no other match falls in $[\![\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1\,, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2]\!]$ during $[\mathbf{a}_\lambda t_0\,, \mathbf{a}_\lambda(t_0 + \kappa^0_{\lambda,\pi})]$.

Then, on $\Omega^{P,T}_{\lambda,\pi}(x_0, t_0)$, we have

$$C^P((\eta^{\lambda,\pi}_t(i))_{t\geq 0, i\in\mathbb{Z}}, (x_0, t_0)) \subset [\![\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1\,, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2]\!].$$

Furthermore, $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda(t_0+\kappa^0_{\lambda,\pi})}(i) \leq 1$ for all $i \in [\![\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1\,, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2]\!]$ and the fire destroys exactly the component $C^P((\eta^{\lambda,\pi}_t(i))_{t\geq 0, i\in\mathbb{Z}}, (x_0, t_0))$.

Indeed, since $\mathbf{m}_\lambda = \mathbf{a}_\lambda \pi(\kappa^0_{\lambda,\pi} - \varepsilon_\lambda)$, on $\Omega^{P,T}_{\lambda,\pi}(x_0, t_0)$, there holds that $T^0_{i_1} \leq \mathbf{a}_\lambda \kappa^0_{\lambda,\pi}$ and $T^0_{i_2} \leq \mathbf{a}_\lambda \kappa^0_{\lambda,\pi}$ (the left front satisfies $i^-_{\mathbf{a}_\lambda \kappa^0_{\lambda,\pi}} \leq i_1$ and the right front satifies $i^+_{\mathbf{a}_\lambda \kappa^0_{\lambda,\pi}} \geq i_2$, thanks to Lemma II.4.2). Consequently,

$$C^P((\eta^{\lambda,\pi}_t(i))_{t\geq 0, i\in\mathbb{Z}}, (x_0, t_0)) \coloneqq [\![\lfloor \mathbf{n}_\lambda x_0 \rfloor + i^g\,, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i^d]\!] \subset [\![\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1\,, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2]\!]$$

where i^g and i^d are defined in Definition II.4.8. Observe now that, by construction, for all $i \in [\![i^g\,, i^d]\!]$

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0 + T^0_i}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i) = 2 = \check{\zeta}^{\lambda,\pi,0}_{T^0_i}(i)$$

and $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0 + T^0_{i^g-1}}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i^g - 1) = 0 = \eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0 + T^0_{i^d+1}}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i^d + 1)$. Recall that on $\Omega^{T,P}_{\lambda,\pi}(x_0, t_0)$, a spark at time $t \in [0\,, \mathbf{a}_\lambda T]$ for the process $(\check{\zeta}^{\lambda,\pi,0}_t(i))_{t\geq 0, i\in\mathbb{Z}}$ has vacant neighbors. Since for all $i \in [\![i^g\,, i^d]\!]$, the processes $(\check{\zeta}^{\lambda,\pi,0}_t(i))_{t\geq 0}$ and $(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0 + t}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i))_{t\geq 0}$ evolve with the same seed processes and the same propagation processes after burning for the first time until $\mathbf{a}_\lambda \kappa^0_{\lambda,\pi}$ and since the zone $[\![i^g\,, i^d]\!]$ is protected by the vacant sites i_1 and i_2, a straightforward observation shows that for all $i \in [\![i^g + 1\,, i^d - 1]\!]$,

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda(t_0+\kappa^0_{\lambda,\pi})}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i) = \check{\zeta}^{\lambda,\pi,0}_{\mathbf{a}_\lambda \kappa^0_{\lambda,\pi}}(i).$$

Observe also that i^g and i^d burn exactly once during $[\mathbf{a}_\lambda t_0 \, , \mathbf{a}_\lambda(t_0 + \kappa^0_{\lambda,\pi})]$ (because the site i^d+1 is vacant at time $T^0_{i^d+1}$ and i^g-1 is vacant at time $T^0_{i^g-1}$ with $T^0_{i^g} \vee T^0_{i^d} \le \mathbf{a}_\lambda \kappa^0_{\lambda,\pi}$). Thus, a site $i \in [\![\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1 \, , \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2]\!] \setminus C^P((\eta^{\lambda,\pi}_t(i))_{t\ge0,i\in\mathbb{Z}}, (x_0,t_0))$ can't be burnt during $[\mathbf{a}_\lambda t_0 \, , \mathbf{a}_\lambda(t_0 + \kappa^0_{\lambda,\pi})]$.

On $\Omega^{P,T}_{\lambda,\pi}(x_0,t_0)$, there is no more burning tree in $[\![-\mathbf{m}_\lambda \, , \mathbf{m}_\lambda]\!] \supset [\![i^g \, , i^d]\!]$ at time $\mathbf{a}_\lambda \kappa^0_{\lambda,\pi}$ for the process $(\check{\zeta}^{\lambda,\pi,0}_t(i))_{t\ge0,i\in\mathbb{Z}}$ (because $\mathbf{m}_\lambda = \mathbf{a}_\lambda \pi(\kappa^0_{\lambda,\pi} - \varepsilon_\lambda)$) and consequently, its also holds true in $[\![\lfloor \mathbf{n}_\lambda x_0 \rfloor + i^g \, , \lfloor \mathbf{n}_\lambda x_0 \rfloor + i^d]\!]$ at time $\mathbf{a}_\lambda(t_0 + \kappa^0_{\lambda,\pi})$ for the process $(\eta^{\lambda,\pi}_t(i))_{t\ge0,i\in\mathbb{Z}}$.

All this implies that, on $\Omega^{P,T}_{\lambda,\pi}(x_0,t_0)$,

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda(t_0+\kappa^0_{\lambda,\pi})}(i) \le 1 \text{ for all } i \in C^P((\eta^{\lambda,\pi}_t(i))_{t\ge0,i\in\mathbb{Z}}, (x_0,t_0))$$

and therefore for all $i \in [\![\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1 \, , \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2]\!]$.

Micro(0)**:** here we focus on the regime $\mathcal{R}(0)$. Let $A, B > 0$ and recall that, for $A > 0$, $\varkappa^A_{\lambda,\pi} = \frac{\mathbf{n}_\lambda A}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda$. Assume that

- ▷ there are $-\mathbf{m}_\lambda < i_1 < 0 < i_2 < \mathbf{m}_\lambda$ such that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1) = \eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2) = 0$ for all $t \in [t_0 \, , t_0 + \varkappa^{A\vee B}_{\lambda,\pi}]$,
- ▷ there is no burning tree in $[\![\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1 \, , \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2]\!]$ at time $\mathbf{a}_\lambda t_0-$,
- ▷ no other match falls in $[\![\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1 \, , \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2]\!]$ during $[\mathbf{a}_\lambda t_0 \, , \mathbf{a}_\lambda(t_0 + \varkappa^{A\vee B}_{\lambda,\pi})]$.

Then, on $\Omega^{P,A,B}_{\lambda,\pi}(x_0,t_0)$, we have

$$C^P((\eta^{\lambda,\pi}_t(i))_{t\ge0,i\in\mathbb{Z}}, (x_0,t_0)) \subset [\![\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1 \, , \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2]\!].$$

Furthermore, $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda(t_0+\varkappa^{A\vee B}_{\lambda,\pi})}(i) \le 1$ for all $i \in [\![\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1 \, , \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2]\!]$ and the fire destroys exactly the zone $C^P((\eta^{\lambda,\pi}_t(i))_{t\ge0,i\in\mathbb{Z}}, (x_0,t_0))$.

Indeed, this can be checked exactly as above (replace $\kappa^0_{\lambda,\pi}$ by $\varkappa^{A\vee B}_{\lambda,\pi}$ and $\Omega^{P,T}_{\lambda,\pi}(x_0,t_0)$ by $\Omega^{P,A,B}_{\lambda,\pi}(x_0,t_0)$).

Micro(∞, z_0)**:** here we focus on the regime $\mathcal{R}(\infty, z_0)$, for some $z_0 \in (0\,,1]$ (in the case $z_0 = 0$, there are only fires of the second kind). Let $0 < z < z_0$ and recall that $\kappa^z_{\lambda,\pi} = \frac{1}{\lambda^z \mathbf{a}_\lambda \pi} + \varepsilon_\lambda$. Assume that

- ▷ there are $-\lfloor \lambda^{-z} \rfloor < i_1 < 0 < i_2 < \lfloor \lambda^{-z} \rfloor$ such that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1) = \eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2) = 0$ for all $t \in [t_0 \, , t_0 + \kappa^z_{\lambda,\pi}]$,
- ▷ there is no burning tree in $[\![\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1 \, , \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2]\!]$ at time $\mathbf{a}_\lambda t_0-$,
- ▷ no other match falls in $[\![\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1 \, , \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2]\!]$ during $[\mathbf{a}_\lambda t_0 \, , \mathbf{a}_\lambda(t_0 + \kappa^z_{\lambda,\pi})]$.

Then, on $\Omega^{P,z}_{\lambda,\pi}(x_0,t_0)$, as in **Micro**$(p)$ above (replace $\kappa^0_{\lambda,\pi}$ by $\kappa^z_{\lambda,\pi}$ and $\Omega^{P,T}_{\lambda,\pi}(x_0,t_0)$ by $\Omega^{P,z}_{\lambda,\pi}(x_0,t_0)$), we have

$$C^P((\eta^{\lambda,\pi}_t(i))_{t\geq 0,i\in\mathbb{Z}},(x_0,t_0)) := [\![\lfloor \mathbf{n}_\lambda x_0\rfloor + i^g\,,\lfloor \mathbf{n}_\lambda x_0\rfloor + i^d]\!] \subset [\![\lfloor \mathbf{n}_\lambda x_0\rfloor + i_1\,,\lfloor \mathbf{n}_\lambda x_0\rfloor + i_2]\!].$$

Furthermore, $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda(t_0+\kappa^z_{\lambda,\pi})}(i)\leq 1$ for all $i\in[\![\lfloor \mathbf{n}_\lambda x_0\rfloor+i_1\,,\lfloor \mathbf{n}_\lambda x_0\rfloor+i_2]\!]$ and the fire destroys exactly the zone $C^P((\eta^{\lambda,\pi}_t(i))_{t\geq 0,i\in\mathbb{Z}},(x_0,t_0))$.

More precisely, on $\Omega^{P,z}_{\lambda,\pi}(x_0,t_0)$, for the process $(\check{\zeta}^{\lambda,\pi,0}_t(i))_{t\geq 0,i\in\mathbb{Z}}$, all site $i\in[\![i^{0,-}_{\mathbf{a}_\lambda\kappa^z_{\lambda,\pi}}+1\,,i^{0,+}_{\mathbf{a}_\lambda\kappa^z_{\lambda,\pi}}-1]\!]$ burns exactly once before $\mathbf{a}_\lambda\kappa^z_{\lambda,\pi}$ whence there is no spark in $[\![i^{0,-}_{\mathbf{a}_\lambda\kappa^z_{\lambda,\pi}}+1\,,i^{0,+}_{\mathbf{a}_\lambda\kappa^z_{\lambda,\pi}}-1]\!]$ at any time $t\in[0\,,\mathbf{a}_\lambda\kappa^z_{\lambda,\pi}]$.

Since, for all $i\in[\![i^g\,,i^d]\!]$, the processes $(\check{\zeta}^{\lambda,\pi,0}_t(i))_{t\geq 0}$ and $(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0+t}(\lfloor \mathbf{n}_\lambda x_0\rfloor+i))_{t\geq 0}$ evolve with the same seed processes and the same propagation processes after burning for the first time until $\mathbf{a}_\lambda\kappa^z_{\lambda,\pi}$, a straightforward observation shows that, for all $t\in[\mathbf{a}_\lambda t_0\,,\mathbf{a}_\lambda(t_0+\kappa^z_{\lambda,\pi})]$, and all $i\in C^P((\eta^{\lambda,\pi}_t(i))_{t\geq 0,i\in\mathbb{Z}},(x_0,t_0))$, for $i\geq\lfloor \mathbf{n}_\lambda x_0\rfloor$, $\eta^{\lambda,\pi}_t(i)$ equals to

$$\begin{cases}\min(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i)+N^S_{t+\mathbf{a}_\lambda t_0-}(i)-N^S_{\mathbf{a}_\lambda t_0}(i),1) & \text{if } \mathbf{a}_\lambda t_0\leq t<\mathbf{a}_\lambda t_0+T^0_{i-\lfloor \mathbf{n}_\lambda x_0\rfloor}\\ 2 & \text{if } \mathbf{a}_\lambda t_0+T^0_{i-\lfloor \mathbf{n}_\lambda x_0\rfloor}\leq t<\mathbf{a}_\lambda t_0+T^0_{i+1-\lfloor \mathbf{n}_\lambda x_0\rfloor}\\ \min(N^S_t(i)-N^S_{T_{i+1-\lfloor \mathbf{n}_\lambda x_0\rfloor}}(i),1) & \text{if } \mathbf{a}_\lambda t_0+T^0_{i+1-\lfloor \mathbf{n}_\lambda x_0\rfloor}\leq t\leq\mathbf{a}_\lambda(t_0+\kappa^z_{\lambda,\pi}),\end{cases}$$

and, for $i\leq\lfloor \mathbf{n}_\lambda x_0\rfloor$, $\eta^{\lambda,\pi}_t(i)$ equals to

$$\begin{cases}\min(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0-}(i)+N^S_{t+\mathbf{a}_\lambda t_0}(i)-N^S_{\mathbf{a}_\lambda t_0}(i),1) & \text{if } \mathbf{a}_\lambda t_0\leq t<\mathbf{a}_\lambda t_0+T^0_{i-\lfloor \mathbf{n}_\lambda x_0\rfloor}\\ 2 & \text{if } \mathbf{a}_\lambda t_0+T^0_{i-\lfloor \mathbf{n}_\lambda x_0\rfloor}\leq t<\mathbf{a}_\lambda t_0+T^0_{i-1-\lfloor \mathbf{n}_\lambda x_0\rfloor}\\ \min(N^S_t(i)-N^S_{T^0_{i-1-\lfloor \mathbf{n}_\lambda x_0\rfloor}}(i),1) & \text{if } \mathbf{a}_\lambda t_0+T^0_{i-1-\lfloor \mathbf{n}_\lambda x_0\rfloor}\leq t\leq\mathbf{a}_\lambda(t_0+\kappa^z_{\lambda,\pi}),\end{cases}$$

Finally, for $i\in[\![\lfloor \mathbf{n}_\lambda x_0\rfloor+i_1\,,\lfloor \mathbf{n}_\lambda x_0\rfloor+i_2]\!]\backslash C^P((\eta^{\lambda,\pi}_t(i))_{t\geq 0,i\in\mathbb{Z}},(x_0,t_0))$ and $t\in[\mathbf{a}_\lambda t_0\,,\mathbf{a}_\lambda(t_0+\kappa^z_{\lambda,\pi})]$, $\eta^{\lambda,\pi}_t(i)$ is nothing but $\min(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i)+N^S_{t+\mathbf{a}_\lambda t_0}(i)-N^S_{\mathbf{a}_\lambda t_0}(i),1)$.

Macroscopic fire:

let $\lambda\in(0\,,1]$ and $\pi\geq 1$. Recall that, for $x>x_0$, $T^0_{\lfloor \mathbf{n}_\lambda x\rfloor-\lfloor \mathbf{n}_\lambda x_0\rfloor}$ is the time needed for the fire starting in $\lfloor \mathbf{n}_\lambda x_0\rfloor$ at time $\mathbf{a}_\lambda t_0$ to reach $\lfloor \mathbf{n}_\lambda x\rfloor$.

Macro(p)**:** here we focus on the regime $\mathcal{R}(p)$, for some $p>0$. On $\Omega^{P,T}_{\lambda,\pi}(x_0,t_0)$, if $0\leq x-x_0\leq(T-t_0-\varepsilon_\lambda)\frac{\mathbf{a}_\lambda\pi}{\mathbf{n}_\lambda}$, there holds that

$$\frac{\mathbf{a}_\lambda t_0+T^0_{\lfloor \mathbf{n}_\lambda x\rfloor-\lfloor \mathbf{n}_\lambda x_0\rfloor}}{\mathbf{a}_\lambda}\in[t_0+\frac{\lfloor \mathbf{n}_\lambda x\rfloor-\lfloor \mathbf{n}_\lambda x_0\rfloor}{\mathbf{a}_\lambda\pi}-\varepsilon_\lambda\,,t_0+\frac{\lfloor \mathbf{n}_\lambda x\rfloor-\lfloor \mathbf{n}_\lambda x_0\rfloor}{\mathbf{a}_\lambda\pi}+\varepsilon_\lambda]$$

and observe that, when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$,

$$[t_0 + \frac{\lfloor \mathbf{n}_\lambda x \rfloor - \lfloor \mathbf{n}_\lambda x_0 \rfloor}{\mathbf{a}_\lambda \pi} - \varepsilon_\lambda \, , t_0 + \frac{\lfloor \mathbf{n}_\lambda x \rfloor - \lfloor \mathbf{n}_\lambda x_0 \rfloor}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda] \simeq \{t_0 + p(x - x_0)\}.$$

This is just a rewriting of Lemma II.4.2.

Macro(0)**:** here we focus on the regime $\mathcal{R}(0)$. On $\Omega_{\lambda,\pi}^{P,A,B}(x_0, t_0)$, for some $B > x - x_0$ and $A > 0$, there holds that

$$\frac{\mathbf{a}_\lambda t_0 + T^0_{\lfloor \mathbf{n}_\lambda x \rfloor - \lfloor \mathbf{n}_\lambda x_0 \rfloor}}{\mathbf{a}_\lambda} \in [t_0 \, , t_0 + \varkappa_{\lambda,\pi}^B]$$

and observe that $[t_0 \, , t_0 + \varkappa_{\lambda,\pi}^B] \simeq \{t_0\}$ when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$.

Besides, assume that

- $\triangleright$ there are $\lfloor \mathbf{n}_\lambda (x_0 - A) \rfloor < i_1 < \lfloor \mathbf{n}_\lambda x_0 \rfloor < i_2 < \lfloor \mathbf{n}_\lambda (x_0 + B) \rfloor$ such that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(i_1) = \eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(i_1) = 0$ for all $s \in [t_0 \, , t_0 + \varkappa_{\lambda,\pi}^{A\vee B}]$,
- $\triangleright$ there is no burning tree in $[\![i_1 \, , i_2]\!]$ at time $\mathbf{a}_\lambda t_0$,
- $\triangleright$ no other match falls in $[\![i_1 \, , i_2]\!]$ during $[\mathbf{a}_\lambda t_0 \, , \mathbf{a}_\lambda (t_0 + \varkappa_{\lambda,\pi}^{A\vee B})]$.

Then, on $\Omega_{\lambda,\pi}^{P,A,B}(x_0, t_0)$, we have

$$C^P((\eta_t^{\lambda,\pi}(i))_{t\geq 0, i\in\mathbb{Z}}, (x_0, t_0)) \subset [\![\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1 \, , \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2]\!].$$

Furthermore, $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda (t_0 + \varkappa_{\lambda,\pi}^{A\vee B})}(i) \leq 1$ for all $i \in [\![\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1 \, , \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2]\!]$ and the fire destroys exactly the zone $C^P((\eta_t^{\lambda,\pi}(i))_{t\geq 0, i\in\mathbb{Z}}, (x_0, t_0))$.

This can be shown exactly as in the case **Micro**(0) (the two statement are very similar).

Macro(∞, z_0)**:** here we focus on fires of second kind in the regime $\mathcal{R}(\infty, z_0)$, for some $z_0 \in [0 \, , 1]$. Let $\gamma \in (0 \, , 1)$, on $\Omega_{\lambda,\pi}^{P,T,\gamma}(x_0, t_0)$, there holds that

$$x_0 - \frac{\mathbf{m}_\lambda^\gamma}{\mathbf{n}_\lambda} \leq \frac{\lfloor \mathbf{n}_\lambda x_0 \rfloor + i^{0,-}_{\mathbf{a}_\lambda T}}{\mathbf{n}_\lambda} \leq x_0 \leq \frac{\lfloor \mathbf{n}_\lambda x_0 \rfloor + 1 + i^{0,+}_{\mathbf{a}_\lambda T}}{\mathbf{n}_\lambda} \leq x_0 + \frac{\mathbf{m}_\lambda^\gamma}{\mathbf{n}_\lambda}$$

and observe that $\mathbf{m}_\lambda^\gamma / \mathbf{n}_\lambda \leq \gamma$: this is just a rewriting of Lemma II.4.4. Thus, since γ can be chosen arbitrarily small, in the regime $\mathcal{R}(\infty, z_0)$, fires have only a local effect.

II.5. Localization of the $(\lambda, \pi)-$FFP

Recall that $\mathbf{a}_\lambda, \mathbf{n}_\lambda$ and $\mathbf{m}_\lambda$ are defined in (III.2.2), (III.2.5) and (II.1.4). For $A > 0$, we set $A_\lambda = \lfloor A\mathbf{n}_\lambda \rfloor$ and $I_A^\lambda = [\![-A_\lambda\,, A_\lambda]\!]$. For $i \in \mathbb{Z}$, we set $i_\lambda = [i/\mathbf{n}_\lambda\,, (i+1)/\mathbf{n}_\lambda)$ and $\varepsilon_\lambda = 1/\mathbf{a}_\lambda^3$.

We first introduce the $(\lambda, \pi, A)-$FFP.

Definition II.5.1. *Let $\lambda \in (0\,, 1], \pi \geq 1$ and $A > 0$ be fixed. For each $i \in I_A^\lambda$, we consider three independent Poisson processes, $N^S(i) = (N_t^S(i))_{t\geq 0}, N^M(i) = (N_t^M(i))_{t\geq 0}$ and $N^P(i) = (N_t^P(i))_{t\geq 0}$ of respective parameters $1, \lambda$ and π, all these processes being independent. Consider a $\{0, 1, 2\}$-valued process $(\eta_t^{\lambda,\pi,A}(i))_{t\geq 0, i\in I_A^\lambda}$ such that a.s., for all $i \in I_A^\lambda$, $(\eta_t^{\lambda,\pi,A}(i))_{t\geq 0}$ is càdlàg. We say that $(\eta_t^{\lambda,\pi,A}(i))_{t\geq 0, i\in I_A^\lambda}$ is a $(\lambda, \pi, A)-$FFP if a.s., for all $i \in I_A^\lambda$, all $t \geq 0$*

$$
\begin{aligned}
\eta_t^{\lambda,\pi,A}(i) = &\int_0^t \mathbf{1}_{\{\eta_{s-}^{\lambda,\pi,A}(i)=0\}}\, \mathrm{d}N_s^S(i) + \int_0^t \mathbf{1}_{\{\eta_{s-}^{\lambda,\pi,A}(i)=1\}}\, \mathrm{d}N_s^M(i) \\
&+ \int_0^t \mathbf{1}_{\{\eta_{s-}^{\lambda,\pi,A}(i+1)=2, \eta_{s-}^{\lambda,\pi,A}(i)=1\}}\, \mathrm{d}N_s^P(i+1) \\
&+ \int_0^t \mathbf{1}_{\{\eta_{s-}^{\lambda,\pi,A}(i-1)=2, \eta_{s-}^{\lambda,\pi,A}(i)=1\}}\, \mathrm{d}N_s^P(i-1) \\
&- 2\int_0^t \mathbf{1}_{\{\eta_{s-}^{\lambda,\pi,A}(i)=2\}}\, \mathrm{d}N_s^P(i)
\end{aligned}
$$

with the convention $N_t^S(A_\lambda + 1) = N_t^S(-A_\lambda - 1) = 0$ for all $t \geq 0$.

For $\eta \in \{0, 1, 2\}^{I_A^\lambda}$ and $i \in I_A^\lambda$, we define the occupied connected component around i as

$$
C_A(\eta, i) = \begin{cases} \emptyset & \text{if } \eta(i) = 0 \text{ or } 2, \\ [\![l_A(\eta, i)\,, r_A(\eta, i)]\!] & \text{if } \eta(i) = 1, \end{cases}
$$

where

$$
\begin{aligned}
l_A(\eta, i) &= (-A_\lambda) \vee (\sup\{k < i :\ \eta(k) = 0 \text{ or } 2\} + 1), \\
r_A(\eta, i) &= A_\lambda \wedge (\inf\{k > i :\ \eta(k) = 0 \text{ or } 2\} - 1)\,.
\end{aligned}
$$

For $x \in [-A\,, A]$ and $t \geq 0$, we also introduce

$$
D_t^{\lambda,\pi,A}(x) = \frac{1}{\mathbf{n}_\lambda} C_A\left(\eta_{\mathbf{a}_\lambda t}^{\lambda,\pi,A}, \lfloor \mathbf{n}_\lambda x \rfloor\right) \subset [-A_\lambda/\mathbf{n}_\lambda\,, A_\lambda/\mathbf{n}_\lambda] \simeq [-A\,, A], \tag{II.5.1}
$$

$$
K_t^{\lambda,\pi,A}(x) = \frac{\left|\left\{i \in [\![\lfloor \mathbf{n}_\lambda x \rfloor - \mathbf{m}_\lambda\,, \lfloor \mathbf{n}_\lambda x \rfloor + \mathbf{m}_\lambda]\!] \cap I_A^\lambda : \eta_{\mathbf{a}_\lambda t}^{\lambda,\pi,A}(i) = 1\right\}\right|}{|[\![\lfloor \mathbf{n}_\lambda x \rfloor - \mathbf{m}_\lambda\,, \lfloor \mathbf{n}_\lambda x \rfloor + \mathbf{m}_\lambda]\!] \cap I_A^\lambda|} \in [0\,, 1], \tag{II.5.2}
$$

$$
Z_t^{\lambda,\pi,A}(x) = \frac{-\log\left(1 - K_t^{\lambda,\pi,A}(x)\right)}{\log(1/\lambda)} \wedge 1 \in [0\,, 1]. \tag{II.5.3}
$$

We now give a discrete version of Proposition II.3.5. Recall Definition II.4.1.

Proposition II.5.2. *Let $T > 0, \lambda \in (0\,,1]$ and $\pi \geq 1$. For each $i \in \mathbb{Z}$, we consider three Poisson processes $N^S(i) = (N^S_t(i))_{t\geq 0}, N^M(i) = (N^M_t(i))_{t\geq 0}$ and $N^P(i) = (N^P_t(i))_{t\geq 0}$ with respective parameters 1, λ and π, all these processes being independent. Let $(\eta^{\lambda,\pi}_t(i))_{t\geq 0, i\in\mathbb{Z}}$ be the corresponding $(\lambda,\pi)-$FFP and for each $A > 0$, let $(\eta^{\lambda,\pi,A}_t(i))_{t\geq 0, i\in I^\lambda_A}$ be the corresponding $(\lambda,\pi,A)-$FFP. There are some constants $\alpha_T > 0$ and $C_T > 0$ such that for all $A \geq 1$, for all (λ,π) sufficiently close to the regime $\mathcal{R}(p)$, for some $p \geq 0$ (or to the regime $\mathcal{R}(\infty, z_0)$, for some $z_0 \in [0\,,1]$),*

$$\mathbb{P}\Big[(\eta^{\lambda,\pi}_t(i))_{t\in[0,\mathbf{a}_\lambda T], i\in I^\lambda_{A/2}} = (\eta^{\lambda,\pi,A}_t(i))_{t\in[0,\mathbf{a}_\lambda T], i\in I^\lambda_{A/2}},$$
$$(Z^{\lambda,\pi}_t(x), D^{\lambda,\pi}_t(x))_{t\in[0,T], x\in[-A/2,A/2]} = (Z^{\lambda,\pi,A}_t(x), D^{\lambda,\pi,A}_t(x))_{t\in[0,T], x\in[-A/2,A/2]}\Big]$$
$$\geq 1 - C_T e^{-\alpha_T A}.$$

Observe that the Proposition II.5.2 holds for the three regimes, with the same scales but for different reasons. We thus distinguish the three regimes. The proof given for $p = 0$ can be adapted in order to work for $p > 0$, as in Proposition II.3.5, but the proof given here for $p > 0$ is much simpler.

Proof in the regime $\mathcal{R}(p)$ for some $p > 0$. Consider the true $(\lambda,\pi)-$FFP $(\eta^{\lambda,\pi}_t(i))_{t\geq 0, i\in\mathbb{Z}}$. It of course suffices to prove the result for A large enough. Temporarily assume that for $a \in \mathbb{R}$, there is an event $\Omega^{\lambda,\pi}_{a,T}$, depending only on the Poisson processes $N^S_t(i), N^M_t(i)$ and $N^P_t(i)$ for $t \in [0\,, \mathbf{a}_\lambda(T+2)]$ and

$$i \in \bar{J}^\lambda_a := [\![\lfloor (a - 1 - 2\frac{T-1}{p})\mathbf{n}_\lambda \rfloor\,, \lfloor (a + 1 + 2\frac{T-1}{p})\mathbf{n}_\lambda \rfloor - 1]\!],$$

such that

(i) on $\Omega^{\lambda,\pi}_{a,T}$, a.s., there are $\iota^+ \colon [0\,, \mathbf{a}_\lambda T] \mapsto \bar{J}^\lambda_a$ non decreasing and $\iota^- \colon [0\,, \mathbf{a}_\lambda T] \mapsto \bar{J}^\lambda_a$ non increasing such that $\eta^{\lambda,\pi}_t(\iota^+_t) = 0$ or 2 and $\eta^{\lambda,\pi}_t(\iota^-_t) = 0$ or 2 for all $t \in [0\,, \mathbf{a}_\lambda T]$,

(ii) there exists $q_T > 0$ such that for all $a \in \mathbb{R}$, we have $\mathbb{P}\left[\Omega^{\lambda,\pi}_{a,T}\right] \geq q_T$, for all (λ,π) sufficiently close to the regime $\mathcal{R}(p)$.

The proof is then concluded using similar argument as Step 3 in the proof of Proposition II.3.5: thanks to point (ii), the probability that there are $-A + 1 + 2\frac{T-1}{p} < a_1 < -A/2 - 1 - 2\frac{T-1}{p}$ and $A/2 + 1 + 2\frac{T-1}{p} < a_2 < A - 1 - 2\frac{T-1}{p}$ with $\Omega^{\lambda,\pi}_{a_1,T}$ and $\Omega^{\lambda,\pi}_{a_2,T}$ realized is easily bounded from below by $1 - C_T e^{-\alpha_T A}$. Next, on this event, a fire starting at the left of $\lfloor (a_1 - 1 - 2\frac{T-1}{p})\mathbf{n}_\lambda \rfloor$ will never cross $\lfloor (a_1 + 1 + 2\frac{T-1}{p})\mathbf{n}_\lambda \rfloor \leq \lfloor -A\mathbf{n}_\lambda/2 \rfloor$ (thanks to ι^+). Same thing holds on the right: a fire starting at the right of $\lfloor (a_2 + 1 + 2\frac{T-1}{p})\mathbf{n}_\lambda \rfloor$ will never cross $\lfloor (a_2 - 1 - 2\frac{T-1}{p})\mathbf{n}_\lambda \rfloor \geq \lfloor A\mathbf{n}_\lambda/2 \rfloor$ (thanks to ι^-). Finally, the clusters $D^{\lambda,\pi}_t(x)$ and $D^{\lambda,\pi,A}_t(x)$ clearly coincide for all $x \in [-\frac{A}{2}\,, \frac{A}{2}]$ and all $t \in [0\,, T]$.

Step 1. Fix some $\alpha > 0$ small enough, say $\alpha = 0.001$. Recall that $\kappa^0_{\lambda,\pi} = \mathbf{m}_\lambda/(\mathbf{a}_\lambda\pi)+\varepsilon_\lambda$.

For $\lambda > 0, \pi \geq 1$ and $a \in \mathbb{R}$, we define the event $\tilde{\Omega}^{\lambda,\pi}_{a,T}$ on which points 1 and 2 below are satisfied:

1. The family of Poisson processes $(N^M_t(i))_{t\in[0,\mathbf{a}_\lambda T], i\in\bar{J}^\lambda_a}$ has exactly 4 marks in $\bar{J}^\lambda_a$, and we call them $\{(X^\lambda_1, T^\lambda_1), (X^\lambda_2, T^\lambda_2), (X^\lambda_3, T^\lambda_3), (X^\lambda_4, T^\lambda_4)\}$, in such a way the match $(X^\lambda_1, T^\lambda_1)$ (resp. $(X^\lambda_2, T^\lambda_2)$) belongs to
$$[\![\lfloor(a - \frac{5}{6} - \frac{T-1}{p})\mathbf{n}_\lambda\rfloor \,,\, \lfloor(a - \frac{2}{3} - \frac{T-1}{p})\mathbf{n}_\lambda\rfloor]\!] \times [\mathbf{a}_\lambda(\frac{3}{4}+\alpha) \,,\, \mathbf{a}_\lambda(1-\alpha)]$$
$$\text{(resp. } [\![\lfloor(a + \frac{2}{3} + \frac{T-1}{p})\mathbf{n}_\lambda\rfloor \,,\, \lfloor(a + \frac{5}{6} + \frac{T-1}{p})\mathbf{n}_\lambda\rfloor]\!] \times [\mathbf{a}_\lambda(\frac{3}{4}+\alpha) \,,\, \mathbf{a}_\lambda(1-\alpha)]),$$
and the match $(X^\lambda_3, T^\lambda_3)$ (resp. $(X^\lambda_4, T^\lambda_4)$) belongs to
$$[\![\lfloor(a - \frac{1}{2} - \frac{T-1}{p})\mathbf{n}_\lambda\rfloor + 1 \,,\, \lfloor(a - \frac{1}{3} - \frac{T-1}{p})\mathbf{n}_\lambda\rfloor]\!] \times [\mathbf{a}_\lambda(1+\alpha) \,,\, \mathbf{a}_\lambda(\frac{3}{2}-\alpha)]$$
$$\text{(resp. } [\![\lfloor(a + \frac{1}{3} + \frac{T-1}{p})\mathbf{n}_\lambda\rfloor \,,\, \lfloor(a + \frac{1}{2} + \frac{T-1}{p})\mathbf{n}_\lambda\rfloor - 1]\!] \times [\mathbf{a}_\lambda(1+\alpha) \,,\, \mathbf{a}_\lambda(\frac{3}{2}-\alpha)]).$$

2. The family of Poisson processes $(N^S_t(i))_{t\geq 0, i\in\bar{J}^\lambda_a}$ satisfies
   a) for $k = 1, 2$, for all $i \in [\![X^\lambda_k - \lfloor\lambda^{-3/4}\rfloor \,,\, X^\lambda_k + \lfloor\lambda^{-3/4}\rfloor]\!]$, $N^S_{T^\lambda_k}(i) > 0$;
   b) for $k = 1, 2$, there are $i^k_1 \in [\![X^\lambda_k - \mathbf{m}_\lambda + 1 \,,\, X^\lambda_k - \lfloor\lambda^{-3/4}\rfloor - 1]\!]$ and $i^k_2 \in [\![X^\lambda_k + \lfloor\lambda^{-3/4}\rfloor + 1 \,,\, X^\lambda_k + \mathbf{m}_\lambda - 1]\!]$ such that $N^S_{T^\lambda_k+\mathbf{a}_\lambda\kappa^0_{\lambda,\pi}}(i^k_1) = N^S_{T^\lambda_k+\mathbf{a}_\lambda\kappa^0_{\lambda,\pi}}(i^k_2) = 0$;
   c) for $k = 1, 2$, there is $i^k_3 \in [\![X^\lambda_k - \lfloor\lambda^{-3/4}\rfloor \,,\, X^\lambda_k + \lfloor\lambda^{-3/4}\rfloor]\!]$ such that $N^S_{3\mathbf{a}_\lambda/2}(i^k_3) - N^S_{T^\lambda_k}(i^k_3) = 0$;
   d) for all $i \in [\![\lfloor(a - 1 - \frac{T-1}{p})\mathbf{n}_\lambda\rfloor \,,\, \lfloor(a + 1 + \frac{T-1}{p})\mathbf{n}_\lambda\rfloor]\!]$, $N^S_{\mathbf{a}_\lambda(1+\alpha)}(i) > 0$.

We now introduce the event $\Omega^P_{a,T}(\lambda, \pi)$ on which all these four fires propagate at the good speed
$$\Omega^P_{a,T}(\lambda, \pi) = \bigcap_{i=1}^4 \Omega^{P,T}_{\lambda,\pi}\left(\frac{X^\lambda_i}{\mathbf{n}_\lambda}, \frac{T^\lambda_i}{\mathbf{a}_\lambda}\right)$$
recall Definition II.4.7. We finally set
$$\Omega^{\lambda,\pi}_{a,T} = \tilde{\Omega}^{\lambda,\pi}_{a,T} \cap \Omega^P_{a,T}(\lambda, \pi).$$

Step 2. We now prove that on $\Omega^{\lambda,\pi}_{a,T}$, as soon as $\kappa^0_{\lambda,\pi} \leq \alpha/2$, there exist $(\iota^+_t)_{t\in[0,\mathbf{a}_\lambda T]}$ and $(\iota^-_t)_{t\in[0,\mathbf{a}_\lambda T]}$ which satisfy (i).

Indeed, sites i^1_1 and i^1_2 are vacant until $T^\lambda_1 + \mathbf{a}_\lambda\kappa^0_{\lambda,\pi}$ because we start from an vacant initial configuration and 2-(b). On the one hand, they protect the zone $[\![i^1_1 + 1 \,,\, i^1_2 - 1]\!]$ and thus, the zone $[\![X^\lambda_1 - \lfloor\lambda^{-3/4}\rfloor \,,\, X^\lambda_1 + \lfloor\lambda^{-3/4}\rfloor]\!] \subset [\![i^1_1 + 1 \,,\, i^1_2 - 1]\!]$ is completely filled at time T^λ_1-, thanks to 2-(a). On the other hand, on $\Omega^{P,T}_{\lambda,\pi}(X^\lambda_1/\mathbf{n}_\lambda, T^\lambda_1/\mathbf{a}_\lambda)$, as seen in **Micro**$(p)$ in Subsection II.4.4,

- $\triangleright$ the match falling on X_1^λ at time T_1^λ destroys entirely the zone $[\![X_1^\lambda - \lfloor \lambda^{-3/4} \rfloor \,, X_1^\lambda + \lfloor \lambda^{-3/4} \rfloor]\!]$ before $T_1^\lambda + \mathbf{a}_\lambda \kappa_{\lambda,\pi}^0$ (it is still protected by i_1^1 and i_1^2),
- $\triangleright$ the fire does not affect the zone outside $[\![i_1^1 \,, i_2^1]\!]$,
- $\triangleright$ there is no more burning tree in the zone $[\![i_1^1 \,, i_2^1]\!]$ at time $T_1^\lambda + \mathbf{a}_\lambda \kappa_{\lambda,\pi}^0$.

Then, since no seed fall on i_3^1 during $[T_1^\lambda \,, 3\mathbf{a}_\lambda/2)$, i_3^1 remains vacant since it burnt (this happened between T_1^λ and $T_1^\lambda + \mathbf{a}_\lambda \kappa_{\lambda,\pi}^0$) until time $3\mathbf{a}_\lambda/2$, thanks to 2-(c).

Remark that same considerations holds around X_2^λ: the match falling in X_2^λ at time T_2^λ doesn't affect the zone outside $[\![i_1^2 \,, i_2^2]\!]$ (because they remain vacant until time $T_2^\lambda + \mathbf{a}_\lambda \kappa_{\lambda,\pi}^0$), and i_3^2 remains vacant during $[T_2^\lambda + \mathbf{a}_\lambda \kappa_{\lambda,\pi}^0 \,, 3\mathbf{a}_\lambda/2)$.

All this implies that the zone $[\![\lfloor (a - \frac{1}{2} - \frac{T-1}{p})\mathbf{n}_\lambda \rfloor \,, \lfloor (a + \frac{1}{2} + \frac{T-1}{p})\mathbf{n}_\lambda \rfloor]\!]$ is protected from all the fire until $3\mathbf{a}_\lambda/2$ (except possibles those falling at $(X_3^\lambda, T_3^\lambda)$ and $(X_4^\lambda, T_4^\lambda)$). Thus, thanks to 2-(d), the zone $[\![\lfloor (a - \frac{1}{2} - \frac{T-1}{p})\mathbf{n}_\lambda \rfloor \,, \lfloor (a + \frac{1}{2} + \frac{T-1}{p})\mathbf{n}_\lambda \rfloor]\!]$ is completely occupied at time $\mathbf{a}_\lambda(1+\alpha)$.

Since now, on $\Omega_{\lambda,\pi}^{P,T}\left(\frac{X_3^\lambda}{\mathbf{n}_\lambda}, \frac{T_3^\lambda}{\mathbf{a}_\lambda}\right)$, the right front $(i_t^{3,+})_{t\geq 0}$ of the fire ignited at $(X_3^\lambda/\mathbf{n}_\lambda, T_3^\lambda/\mathbf{a}_\lambda)$ statisfies

$$i^{3,+}_{\mathbf{a}_\lambda T - T_3^\lambda} \leq \pi(\mathbf{a}_\lambda T - T_3^\lambda + \mathbf{a}_\lambda \varepsilon_\lambda) \leq \mathbf{a}_\lambda \pi (T - 1 - \alpha + \varepsilon_\lambda),$$

recall Lemma II.4.2, then $i^{3,+}_{\mathbf{a}_\lambda T - T_3^\lambda} \leq (T-1)\frac{\mathbf{n}_\lambda}{p}$ as soon as $\left|\frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} - p\right| \leq p\frac{\alpha}{2(T-1)}$ (recall that $2\varepsilon < \alpha$). This in particular implies that

$$X_3^\lambda + i^{3,+}_{\mathbf{a}_\lambda T - T_3^\lambda} \leq \lfloor (a - \frac{1}{3} - \frac{T-1}{p})\mathbf{n}_\lambda \rfloor + (T-1)\frac{\mathbf{n}_\lambda}{p} < \lfloor \mathbf{n}_\lambda a \rfloor.$$

Similarly, on $\Omega_{\lambda,\pi}^{P,T}\left(\frac{X_4^\lambda}{\mathbf{n}_\lambda}, \frac{T_4^\lambda}{\mathbf{a}_\lambda}\right)$ and for $\left|\frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} - p\right| \leq p\frac{\alpha}{2(T-1)}$, we clearly have

$$\lfloor \mathbf{n}_\lambda a \rfloor < \lfloor (a + \frac{1}{3} + \frac{T-1}{p})\mathbf{n}_\lambda \rfloor - (T-1)\frac{\mathbf{n}_\lambda}{p} \leq X_4^\lambda + i^{4,-}_{\mathbf{a}_\lambda T - T_4^\lambda}.$$

We easily deduce that for all $t \in [0 \,, \mathbf{a}_\lambda T - T_3^\lambda]$, $\eta^{\lambda,\pi}_{t+T_3^\lambda}(X_3^\lambda + i_t^{3,+}) = 2$ and for all $t \in [0 \,, \mathbf{a}_\lambda T - T_4^\lambda]$, $\eta^{\lambda,\pi}_{t+T_4^\lambda}(X_4^\lambda + i_t^{4,-}) = 2$.

Finally, we set, for all $t \in [0 \,, \mathbf{a}_\lambda T]$

$$\iota_t^+ = \begin{cases} i_1^1 & \text{if } 0 \leq t < T_1^\lambda + \kappa_{\lambda,\pi}^0, \\ i_3^1 & \text{if } T_1^\lambda + \kappa_{\lambda,\pi}^0 \leq t < T_3^\lambda, \\ X_3^\lambda + i^{3,+}_{t-T_3^\lambda} & \text{if } T_3^\lambda \leq t \leq \mathbf{a}_\lambda T. \end{cases}$$

Clearly, $(\iota_t^+)_{t\in[0,\mathbf{a}_\lambda T]}$ is non decreasing, $\eta_s^{\lambda,\pi}(\iota_s^+)$ is 0 until T_3^λ and 2 between T_3^λ and $\mathbf{a}_\lambda T$.

Similarly, we can choose

$$\iota_t^- = \begin{cases} i_2^2 & \text{if } 0 \le t < T_2^\lambda + \kappa_{\lambda,\pi}^0, \\ i_3^2 & \text{if } T_2^\lambda + \kappa_{\lambda,\pi}^0 \le t < T_4^\lambda, \\ X_4^\lambda + i_{t-T_4^\lambda}^{4,-} & \text{if } T_4^\lambda \le t \le \mathbf{a}_\lambda T. \end{cases}$$

Clearly, $(\iota_t^-)_{t\in[0,\mathbf{a}_\lambda T]}$ is non increasing, $\eta_s^{\lambda,\pi}(\iota_s^-)$ is 0 until T_4^λ and 2 between T_4^λ and $\mathbf{a}_\lambda T$.

Step 3. We now prove (ii). The quantity $\mathbb{P}\left[\Omega_{a,T}^{\lambda,\pi}\right]$ does obviously not depend on $a \in \mathbb{R}$ by spatial invariance. Then, we observe that we can construct N^M by using a Poisson measure π_M on $\mathbb{R} \times [0,\infty)$ with intensity measure $\mathrm{d}x\,\mathrm{d}t$, independent of N^S and N^P, by setting, for all $i \in \mathbb{Z}$,

$$N_t^M(i) = \pi_M(i_\lambda \times [0, t/\mathbf{a}_\lambda]).$$

Hence, the event on which N^M satisifies 1. contains the event $\Omega_{a,T}^M$ on which π_M has exactly 4 marks in $[a-1-2\frac{T-1}{p}, a+1+2\frac{T-1}{p}] \times [0,T]$, which can be called $(X_1,T_1), (X_2,T_2), (X_3,T_3)$ and (X_4,T_4) in such a way (X_1,T_1) (resp. (X_2,T_2)) belongs to

$$[a - \frac{5}{6} - \frac{T-1}{p} + \alpha, a - \frac{2}{3} - \frac{T-1}{p} - \alpha] \times [\frac{3}{4} + \alpha, 1-\alpha]$$
$$(\text{resp. } [a + \frac{2}{3} + \frac{T-1}{p} + \alpha, a + \frac{5}{6} + \frac{T-1}{p} - \alpha] \times [\frac{3}{4} + \alpha, 1-\alpha]),$$

and (X_3,T_3) (resp. (X_4,T_4)) belongs to

$$[a - \frac{1}{2} - \frac{T-1}{p} + \alpha, a - \frac{1}{3} - \frac{T-1}{p} - \alpha] \times [1+\alpha, \frac{3}{2} - \alpha]$$
$$(\text{resp. } [a + \frac{1}{3} + \frac{T-1}{p} + \alpha, a + \frac{1}{2} + \frac{T-1}{p} - \alpha] \times [1+\alpha, \frac{3}{2} - \alpha]).$$

Clearly, the probability $\mathbb{P}\left[\Omega_{a,T}^M\right]$ does not depend on a nor on λ and π and is positive. We then define $q_T > 0$ by

$$\mathbb{P}\left[\Omega_{a,T}^M\right] = 2q_T. \quad (\star)$$

We then use basic consideration on i.i.d. Poisson processes with rate 1 (we write $\mathbb{P}_M$ for the conditional probability w.r.t. π_M) to show that point 2. occurs with high probability.

- For $k = 1, 2$, we have $T_k^\lambda \ge \mathbf{a}_\lambda(3/4+\alpha)$ and

  $$\mathbb{P}_M\left[\forall i \in [\![X_k^\lambda - \lfloor\lambda^{-3/4}\rfloor, X_k^\lambda + \lfloor\lambda^{-3/4}\rfloor]\!], N_{T_k^\lambda}^S(i) > 0\right] \ge (1-\lambda^{3/4+\alpha})^{2\lfloor\lambda^{-3/4}\rfloor+1}$$

  which tends to 1 when $\lambda \to 0$.

- For $k=1,2$, we have $T_k^\lambda + \mathbf{a}_\lambda \kappa^0_{\lambda,\pi} \le \mathbf{a}_\lambda(1-\alpha/2)$ (recall that $\kappa^0_{\lambda,\pi} \le \alpha/2$) and

$$\mathbb{P}_M\left[\exists i_2^k \in [\![X_k^\lambda + \lfloor \lambda^{-3/4}\rfloor + 1 \,, X_k^\lambda + \mathbf{m}_\lambda - 1]\!], N^S_{T_k^\lambda + \mathbf{a}_\lambda\kappa^0_{\lambda,\pi}}(i_j^k) = 0\right] \\ \ge 1 - (1-\lambda^{1-\alpha/2})^{\mathbf{m}_\lambda - \lfloor \lambda^{-3/4}\rfloor - 1}$$

  which tends to 1 when $\lambda \to 0$ (and similar computation for i_1^k).

- For $k=1,2$, we have $T_k^\lambda \ge \mathbf{a}_\lambda(3/4+\alpha)$ and

$$\mathbb{P}_M\left[\exists i \in [\![X_k^\lambda - \lfloor \lambda^{-3/4}\rfloor \,, X_k^\lambda + \lfloor \lambda^{-3/4}\rfloor]\!], N^S_{3\mathbf{a}_\lambda/2}(i) - N^S_{T_k^\lambda}(i) = 0\right] \\ \ge 1 - (1-\lambda^{3/4-\alpha})^{2\lfloor \lambda^{-3/4}\rfloor + 1}$$

  which tends to 1 when $\lambda \to 0$;

- Finally,

$$\mathbb{P}_M\left[\forall i \in [\![\lfloor (a-1-\frac{T-1}{p})\mathbf{n}_\lambda\rfloor \,, \lfloor (a+1+\frac{T-1}{p})\mathbf{n}_\lambda\rfloor]\!], N^S_{\mathbf{a}_\lambda(1+\alpha)}(i) > 0\right] \\ = (1-\lambda^{1+\alpha})^{(2+2\frac{T-1}{p})\mathbf{n}_\lambda}$$

  which tends also to 1 when $\lambda \to 0$.

Next, since π_M is independent of the processes family $(N_t^S(i))_{i\in\mathbb{Z},t\ge 0}$ and $(N_t^P(i))_{i\in\mathbb{Z},t\ge 0}$, Lemma II.4.2 directly imply that, for all $k=1,\dots,4$, $\mathbb{P}_M\left[\Omega^{P,T}_{\lambda,\pi}(X_k,T_k)\right]$ tends to 1 when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$.

All this, together with $(\star)$, implies that $\mathbb{P}\left[\Omega^{\lambda,\pi}_{a,T}\right] \ge q_T > 0$ for all (λ,π) sufficiently close to the regime $\mathcal{R}(p)$.

In the end, for all (λ,π) sufficiently close to the regime $\mathcal{R}(p)$, the event $\Omega^{\lambda,\pi}_{a,T}$ depend only on the Poisson processes $N_t^S(i), N_t^M(i)$ and $N_t^P(i)$ for $t \in [0\,,\mathbf{a}_\lambda(T+2)]$ and $i \in \bar{J}_a^\lambda$. This suffices to conclude the proof. $\square$

Proof in the regime $\mathcal{R}(\infty,z_0)$. Let us fix $z_0 \in [0\,,1]$. Consider the true (λ,π)−FFP $(\eta_t^{\lambda,\pi}(i))_{t\ge 0, i\in\mathbb{Z}}$. We introduce

$$J_a^\lambda = [\![\lfloor a\mathbf{n}_\lambda\rfloor \,, \lfloor (a+1)\mathbf{n}_\lambda\rfloor - 1]\!].$$

As above, for $a \in \mathbb{R}$, we are going to construct an event $\Omega^{\lambda,\pi}_{a,T}$ depending only on the Poisson processes $N_t^S(i), N_t^M(i)$ and $N_t^P(i)$ for $t \in [0\,,\mathbf{a}_\lambda(T+2)]$ and $i \in J_a^\lambda$ such that

(i) on $\Omega^{\lambda,\pi}_{a,T}$, there exists $\iota^+ : [0\,,\mathbf{a}_\lambda T] \mapsto J_a^\lambda$ non decreasing and $\iota^- : [0\,,\mathbf{a}_\lambda T] \mapsto J_a^\lambda$ non increasing such that $\eta_t^{\lambda,\pi}(\iota_t^+) = 0$ or 2 and $\eta_t^{\lambda,\pi}(\iota_t^-) = 0$ or 2 for all $t \in [0\,,\mathbf{a}_\lambda T]$,

(ii) there exists $q_T > 0$ such that for all $a \in \mathbb{R}$, we have $\mathbb{P}\left[\Omega^{\lambda,\pi}_{a,T}\right] \ge q_T$ for all (λ,π) sufficiently close to the regime $\mathcal{R}(\infty,z_0)$.

The proof is then concluded as previously. We divide the proof in two cases.

Case 1: $z_0 \in [0\,,1)$. We fix $\alpha = 0.001$ and $\gamma \in (0\,,(1-z_0)/4)$. Recall that

$$\mathbf{m}_\lambda^\gamma = \left\lfloor \frac{\gamma}{\lambda^{\gamma+(1-\gamma)z_0}\mathbf{a}_\lambda} \right\rfloor \ll \mathbf{m}_\lambda \ll \mathbf{n}_\lambda.$$

Step 1. For $\lambda > 0, \pi \geq 1$ and $a \in \mathbb{R}$, we define the event $\tilde{\Omega}_{a,T}^{\lambda,\pi}$ on which points 1 and 2 below are satisfied:

1. The family of Poisson processes $(N_t^M(i))_{t\in[0,\mathbf{a}_\lambda T],i\in J_a^\lambda}$ has exactly 2 marks in J_a^λ, and we call them $(X_1^\lambda, T_1^\lambda), (X_2^\lambda, T_2^\lambda)$, in such a way that

$$(X_1^\lambda, T_1^\lambda) \in [\![\lfloor a\mathbf{n}_\lambda \rfloor + \mathbf{m}_\lambda\,, \lfloor (a+\frac{1}{2})\mathbf{n}_\lambda \rfloor - \mathbf{m}_\lambda - 1]\!] \times [\mathbf{a}_\lambda(z_0+2\gamma)\,, \mathbf{a}_\lambda(1-\gamma)],$$

$$(X_2^\lambda, T_2^\lambda) \in [\![\lfloor (a+\frac{1}{2})\mathbf{n}_\lambda \rfloor + \mathbf{m}_\lambda\,, \lfloor (a+1)\mathbf{n}_\lambda \rfloor - \mathbf{m}_\lambda - 1]\!] \times [\mathbf{a}_\lambda(z_0+2\gamma)\,, \mathbf{a}_\lambda(1-\gamma)].$$

2. The family of Poisson processes $(N_t^S(i))_{t\geq 0,i\in J_a^\lambda}$ satisfies, for $k=1,2$,
   a) for all $i \in [\![X_k^\lambda - \mathbf{m}_\lambda^\gamma\,, X_k^\lambda + \mathbf{m}_\lambda^\gamma]\!]$, $N_{T_k^\lambda}^S(i) > 0$;
   b) there are $i_1^k \in [\![X_k^\lambda - \mathbf{m}_\lambda + 1\,, X_k^\lambda - \mathbf{m}_\lambda^\gamma - 1]\!]$ and $i_2^k \in [\![X_k^\lambda + \mathbf{m}_\lambda^\gamma + 1\,, X_k^\lambda + \mathbf{m}_\lambda - 1]\!]$ such that $N_{\mathbf{a}_\lambda(1-\gamma)}^S(i_1^k) = N_{\mathbf{a}_\lambda(1-\gamma)}^S(i_2^k) = 0$.

We now introduce the event on which all of these two fires propagate at the correct speed,

$$\Omega_{a,T}^P(\lambda,\pi) = \Omega_{\lambda,\pi}^{P,T,\gamma}\left(\frac{X_1^\lambda}{\mathbf{n}_\lambda}, \frac{T_1^\lambda}{\mathbf{a}_\lambda}\right) \cap \Omega_{\lambda,\pi}^{P,T,\gamma}\left(\frac{X_2^\lambda}{\mathbf{n}_\lambda}, \frac{T_2^\lambda}{\mathbf{a}_\lambda}\right).$$

We finally set

$$\Omega_{a,T}^{\lambda,\pi} = \tilde{\Omega}_{a,T}^{\lambda,\pi} \cap \Omega_{a,T}^P(\lambda,\pi).$$

Step 2. We now prove that on $\Omega_{a,T}^{\lambda,\pi}$, (i) holds.

For $k=1,2$, thanks to 2-(b), the sites i_1^k and i_2^k remain vacant until $\mathbf{a}_\lambda(1-\gamma) > T_k^\lambda$. Thus, no fire can affect the zone $[\![X_k^\lambda - \mathbf{m}_\lambda^\gamma\,, X_k^\lambda + \mathbf{m}_\lambda^\gamma]\!]$ during $[0\,, \mathbf{a}_\lambda(1-\gamma)]$. Hence, the zone $[\![X_k^\lambda - \mathbf{m}_\lambda^\gamma\,, X_k^\lambda + \mathbf{m}_\lambda^\gamma]\!]$ is completely filled at time $T_k^\lambda-$, thanks to 2-(a). On $\Omega_{\lambda,\pi}^{P,T,\gamma}\left(\frac{X_k^\lambda}{\mathbf{n}_\lambda}, \frac{T_k^\lambda}{\mathbf{a}_\lambda}\right) \subset \Omega_{a,T}^P(\lambda,\pi)$, the fire starting in X_k^λ at time T_k^λ does not affect the zone outside $[\![X_k^\lambda - \mathbf{m}_\lambda^\gamma\,, X_k^\lambda + \mathbf{m}_\lambda^\gamma]\!]$ during $[0\,, \mathbf{a}_\lambda T]$, recall $\mathbf{Macro}(\infty, z_0)$ in Subsection II.4.4. Since $X_2^\lambda - X_1^\lambda \geq 2\mathbf{m}_\lambda \geq 2\mathbf{m}_\lambda^\gamma + 1$, we deduce that $\eta_s^{\lambda,\pi}(X_1^\lambda + i_{s-T_1^\lambda}^{1,+}) = 2$ for all $s \in [T_1^\lambda\,, \mathbf{a}_\lambda T]$ and $\eta_s^{\lambda,\pi}(X_2^\lambda + i_{s-T_2^\lambda}^{2,-}) = 2$ for all $s \in [T_2^\lambda\,, \mathbf{a}_\lambda T]$.

Finally, we set, for all $t \in [0\,, \mathbf{a}_\lambda T]$

$$\iota_t^+ = \begin{cases} i_1^1 & \text{if } 0 \leq t < T_1^\lambda, \\ X_1^\lambda + i_{t-T_3^\lambda}^{1,+} & \text{if } T_1^\lambda \leq t \leq \mathbf{a}_\lambda T. \end{cases}$$

The process $(\iota_t^+)_{t\in[0,\mathbf{a}_\lambda T]}$ is non decreasing, $\eta_s^{\lambda,\pi}(\iota_s^+)$ is 0 for $s \in [0\,, T_1^\lambda)$ and 2 for $s \in [T_1^\lambda\,, \mathbf{a}_\lambda T]$.

Similarly, we set for all $t \in [0\,, \mathbf{a}_\lambda T]$,

$$\iota_t^- = \begin{cases} i_2^2 & \text{if } 0 \le t < T_2^\lambda, \\ X_2^\lambda + i_{t-T_2^\lambda}^{2,-} & \text{if } T_2^\lambda \le t \le \mathbf{a}_\lambda T, \end{cases}$$

which also satisfies the requirements.

Step 3. The event $\Omega_{a,T}^{\lambda,\pi}$ also satisfies point (ii).

Indeed, the quantity $\mathbb{P}\left[\Omega_{a,T}^{\lambda,\pi}\right]$ does obviously not depend on $a \in \mathbb{R}$ by spatial invariance. As previously, we can construct N^M by using a Poisson measure π_M on $\mathbb{R} \times [0\,, \infty)$ with intensity measure $\mathrm{d}x\,\mathrm{d}t$, independent of N^S and N^P, by setting, for all $i \in \mathbb{Z}$,

$$N_t^M(i) = \pi_M(i_\lambda \times [0\,, t/\mathbf{a}_\lambda]).$$

Hence, the event on which N^M satisifies 1. contains the event $\Omega_{a,T}^M$ on which π_M has exactly 2 marks in $[a\,, a+1] \times [0\,, T]$, which can be called (X_1, T_1) and (X_2, T_2) such that (remark that $\gamma < 1/4$)

$$\begin{aligned} (X_1, T_1) &\in [a + \gamma\,, a + \frac{1}{2} - \gamma] \times [z_0 + 2\gamma\,, 1 - \gamma] \\ \text{and } (X_2, T_2) &\in [a + \frac{1}{2} + \gamma\,, a + 1 - \gamma] \times [z_0 + 2\gamma\,, 1 - \gamma]. \end{aligned}$$

Clearly, the probability $\mathbb{P}\left[\Omega_{a,T}^M\right]$ does not depend on a nor on λ and π and is positive. We then define $q_T > 0$ by

$$\mathbb{P}\left[\Omega_{a,T}^M\right] = 2q_T. \quad (\star)$$

We then use basic considerations on i.i.d. Poisson processes with rate 1 (we write $\mathbb{P}_M$ for the conditional probability w.r.t. π_M) to show that point 2. occurs with high probability.

- For $k = 1, 2$, we have $T_k^\lambda \ge \mathbf{a}_\lambda(z_0 + 2\gamma)$ and

$$\begin{aligned} \mathbb{P}_M\left[\forall i \in [\![X_k^\lambda - \mathbf{m}_\lambda^\gamma\,, X_k^\lambda + \mathbf{m}_\lambda^\gamma]\!], N_{T_k^\lambda}^S(i) > 0\right] &\ge (1 - \lambda^{z_0+2\gamma})^{2\mathbf{m}_\lambda^\gamma+1} \\ &\simeq \exp(-\lambda^{z_0+2\gamma}\frac{\gamma\lambda^{-\gamma-(1-\gamma)z_0}}{\mathbf{a}_\lambda}) = \exp(-\gamma\frac{\lambda^{\gamma(z_0+1)}}{\mathbf{a}_\lambda}) \end{aligned}$$

  which tends to 1 when $\lambda \to 0$.

- For $k = 1, 2$, we have

$$\mathbb{P}_M\left[\exists i_2^k \in [\![X_k^\lambda + \mathbf{m}_\lambda^\gamma + 1\,, X_k^\lambda + \mathbf{m}_\lambda - 1]\!], N_{\mathbf{a}_\lambda(1-\gamma)}^S(i_2^k) = 0\right] = 1-(1-\lambda^{1-\gamma})^{\mathbf{m}_\lambda - \mathbf{m}_\lambda^\gamma - 1}$$

  which tends to 1 when $\lambda \to 0$, because $\mathbf{m}_\lambda^\gamma \ll \mathbf{m}_\lambda$ and $\lambda^{1-\gamma} \ll \mathbf{m}_\lambda$ (similar computation holds for i_1^k).

Finally, since π_M is independent of the processes family $(N_t^S(i))_{t\geq 0, i\in\mathbb{Z}}$ and $(N_t^P(i))_{t\geq 0, i\in\mathbb{Z}}$, Lemma II.4.2 directly imply that, for all $k=1,2$, $\mathbb{P}_M\left[\Omega_{\lambda,\pi}^{P,T}(X_k,T_k)\right]$ tends to 1 when $\lambda\to 0$ and $\pi\to\infty$ in the regime $\mathcal{R}(\infty,z_0)$.

All this, together with $(\star)$, implies that $\mathbb{P}\left[\Omega_{a,T}^{\lambda,\pi}\right]\geq q_T>0$ for all (λ,π) sufficiently close to the regime $\mathcal{R}(\infty,z_0)$.

In the end, for all (λ,π) sufficiently close to the regime $\mathcal{R}(\infty,z_0)$, the event $\Omega_{a,T}^{\lambda,\pi}$ depend only on the Poisson processes $N_t^S(i)$, $N_t^M(i)$ and $N_t^P(i)$ for $t\in[0\,,\mathbf{a}_\lambda(T+2)]$ and $i\in J_a^\lambda$. This suffices to conclude the proof in the case $z_0\in[0\,,1)$.

Case 2: $z_0=1$. Fix some $\alpha>0$ small enough, say $\alpha=0.001$. Recall that

$$\kappa_{\lambda,\pi}^{1-\alpha}=\frac{1}{\lambda^{1-\alpha}\mathbf{a}_\lambda\pi}+\varepsilon_\lambda$$

and assume that $\kappa_{\lambda,\pi}^{1-\alpha}<\alpha$ (for (λ,π) sufficiently close to the regime $\mathcal{R}(\infty,1)$). We first define the event $\tilde{\Omega}_{a,T}^{\lambda,\pi}$ on which points 1 and 2 below are satisfied:

1. The family of Poisson processes $(N_t^M(i))_{t\in[0,\mathbf{a}_\lambda T],i\in J_a^\lambda}$ has exactly 4 marks in J_a^λ, and we call them $(X_k^\lambda,T_k^\lambda)_{k=1,\dots,4}$, in such a way the match $(X_1^\lambda,T_1^\lambda)$ (resp. $(X_2^\lambda,T_2^\lambda)$) belongs to

$$\begin{aligned}&[\![\lfloor(a+\alpha)\mathbf{n}_\lambda\rfloor\,,\lfloor(a+\frac{1}{4}-\alpha)\mathbf{n}_\lambda\rfloor]\!]\times[\mathbf{a}_\lambda(\frac{3}{4}+\alpha)\,,\mathbf{a}_\lambda(1-2\alpha)]\\ \text{(resp. }&[\![\lfloor(a+\frac{3}{4}+\alpha)\mathbf{n}_\lambda\rfloor\,,\lfloor(a+1-\alpha)\mathbf{n}_\lambda\rfloor]\!]\times[\mathbf{a}_\lambda(\frac{3}{4}+\alpha)\,,\mathbf{a}_\lambda(1-2\alpha)]),\end{aligned}$$

   and the match $(X_3^\lambda,T_3^\lambda)$ (resp. $(X_4^\lambda,T_4^\lambda)$) belongs to

$$\begin{aligned}&[\![\lfloor(a+\frac{1}{4}+\alpha)\mathbf{n}_\lambda\rfloor\,,\lfloor(a+\frac{1}{2}-\alpha)\mathbf{n}_\lambda\rfloor]\!]\times[\mathbf{a}_\lambda(1+\alpha)\,,\mathbf{a}_\lambda(\frac{5}{4}-2\alpha)]\\ \text{(resp. }&[\![\lfloor(a+\frac{1}{2}+\alpha)\mathbf{n}_\lambda\rfloor\,,\lfloor(a+\frac{3}{4}-\alpha)\mathbf{n}_\lambda\rfloor]\!]\times[\mathbf{a}_\lambda(1+\alpha)\,,\mathbf{a}_\lambda(\frac{5}{4}-2\alpha)]).\end{aligned}$$

2. The family of Poisson processes $(N_t^S(i))_{t\geq 0,i\in J_a^\lambda}$ satisfies,

   a) for $k=1,2$, $\forall i\in[\![X_k^\lambda-\lfloor\lambda^{-3/4}\rfloor\,,X_k^\lambda+\lfloor\lambda^{-3/4}\rfloor]\!]$, $N_{T_k^\lambda}^S(i)>0$;

   b) for $k=1,2$, there are $i_1^k\in[\![X_k^\lambda-\lfloor\lambda^{-(1-\alpha)}\rfloor-1\,,X_k^\lambda]\!]$ and $i_2^k\in[\![X_k^\lambda\,,X_k^\lambda+\lfloor\lambda^{-(1-\alpha)}\rfloor+1]\!]$ such that $N^S_{T_k^\lambda+\mathbf{a}_\lambda\kappa_{\lambda,\pi}^{1-\alpha}}(i_j^k)=0$.

   c) for $k=1,2$, there exists $i_3^k\in[\![X_k^\lambda-\lfloor\lambda^{-3/4}\rfloor\,,X_k^\lambda+\lfloor\lambda^{-3/4}\rfloor]\!]$ such that $N^S_{3\mathbf{a}_\lambda/2}(i_3^k)-N^S_{T_k^\lambda}(i_3^k)=0$;

   d) $\forall i\in[\![\lfloor a\mathbf{n}_\lambda\,,\lfloor(a+1)\mathbf{n}_\lambda\rfloor]\!]$, $N^S_{\mathbf{a}_\lambda(1+\alpha)}(i)>0$.

We now introduce the event on which all these four fires propagate on the good speed

$$\Omega^P_{a,T}(\lambda,\pi)=\Omega^{P,1-\alpha}_{\lambda,\pi}(\frac{X_1^\lambda}{\mathbf{n}_\lambda},\frac{T_1^\lambda}{\mathbf{a}_\lambda})\cap\Omega^{P,1-\alpha}_{\lambda,\pi}(\frac{X_2^\lambda}{\mathbf{n}_\lambda},\frac{T_2^\lambda}{\mathbf{a}_\lambda})\cap\Omega^{P,T,\alpha}_{\lambda,\pi}(\frac{X_3^\lambda}{\mathbf{n}_\lambda},\frac{T_3^\lambda}{\mathbf{a}_\lambda})\cap\Omega^{P,T,\alpha}_{\lambda,\pi}(\frac{X_4^\lambda}{\mathbf{n}_\lambda},\frac{T_4^\lambda}{\mathbf{a}_\lambda}),$$

recall Definition II.4.7.

We finally set

$$\Omega^{\lambda,\pi}_{a,T}=\tilde\Omega^{\lambda,\pi}_{a,T}\cap\Omega^P_{a,T}(\lambda,\pi).$$

We deduce that $\Omega^{\lambda,\pi}_{a,T}$ satisfies (i) as above: the match falling in X_k^λ, for $k=1,2$, destroys at least the zone $[\![X_k^\lambda-\lfloor\lambda^{-3/4}\rfloor\,,X_k^\lambda+\lfloor\lambda^{-3/4}\rfloor]\!]$ (thanks to 2-(a)) but does not affect the zone outside $[\![X_k^\lambda-\lfloor\lambda^{-(1-\alpha)}\rfloor\,,X_k^\lambda+\lfloor\lambda^{-(1-\alpha)}\rfloor]\!]$ (thanks to 2-(b) and recall **Micro**$(\infty,1)$ in Subsection II.4.4). Hence, for $k=1,2$, i_3^k remains vacant from $T_k^\lambda+\mathbf{a}_\lambda\kappa^{1-\alpha}_{\lambda,\pi}$ until $3\mathbf{a}_\lambda/2$. Thus, i_3^1 and i_3^2 protect the zone $[\![\lfloor(a+\frac14-\alpha)\mathbf{n}_\lambda\rfloor\,,\lfloor(a+\frac34-\alpha)\mathbf{n}_\lambda\rfloor]\!]$, which is completely filled at time $\mathbf{a}_\lambda(1+\alpha)$, thanks to 2-(d). As previously, and since fires have only a local effect (recall that $\mathbf{m}^\alpha_\lambda=\lfloor\alpha\mathbf{n}_\lambda\rfloor$), the right front of the fire 3 and the left front of the fire 4 burn until $\mathbf{a}_\lambda T$.

We then can set, for all $t\in[0\,,\mathbf{a}_\lambda T]$

$$\iota_t^+=\begin{cases} i_1^1 & \text{if } 0\le t<T_1^\lambda+\mathbf{a}_\lambda\kappa^{1-\alpha}_{\mathbf{a}_\lambda,\pi},\\ i_3^1 & \text{if } T_1^\lambda+\mathbf{a}_\lambda\kappa^{1-\alpha}_{\mathbf{a}_\lambda,\pi}\le t<T_3^\lambda,\\ X_3^\lambda+i^{3,+}_{t-T_3^\lambda} & \text{if } T_3^\lambda\le t\le\mathbf{a}_\lambda T,\end{cases}$$

and

$$\iota_t^-=\begin{cases} i_2^2 & \text{if } 0\le t<T_2^\lambda+\mathbf{a}_\lambda\kappa^{1-\alpha}_{\mathbf{a}_\lambda,\pi},\\ i_3^2 & \text{if } T_2^\lambda+\mathbf{a}_\lambda\kappa^{1-\alpha}_{\mathbf{a}_\lambda,\pi}\le t<T_4^\lambda,\\ X_4^\lambda+i^{4,-}_{t-T_4^\lambda} & \text{if } T_4^\lambda\le t\le\mathbf{a}_\lambda T.\end{cases}$$

We can check, as usual, that $\mathbb{P}\left[\Omega^{\lambda,\pi}_{a,T}\right]\ge q_T$, for all (λ,π) sufficiently close to the regime $\mathcal{R}(\infty,1)$, where $2q_T$ is the probability that a Poisson measure π_M has exactly 4 marks $(X_k,T_k)_{k=1,\dots,4}$ in $[a\,,a+1]\times[0\,,T]$ in such a way that

$$\begin{aligned}
(X_1,T_1)&\in[a+\alpha\,,a+\frac14-\alpha]\times[\frac34+\alpha\,,1-2\alpha],\\
(X_2,T_2)&\in[a+\frac34+\alpha\,,a+1-\alpha]\times[\frac34+\alpha\,,1-2\alpha],\\
(X_3,T_3)&\in[a+\frac14+\alpha\,,a+\frac12-\alpha]\times[1+\alpha\,,\frac54-2\alpha],\\
(X_4,T_4)&\in[a+\frac12+\alpha\,,a+\frac34-\alpha]\times[1+\alpha\,,\frac54-2\alpha].
\end{aligned}$$

□

Proof in the regime $\mathcal{R}(0)$. We fix $T>0$. It of course suffices to prove the result for A large enough. We consider the true $(\lambda,\pi)-$FFP $(\eta_t^{\lambda,\pi}(i))_{t\ge0,i\in\mathbb{Z}}$ and set $K=\lfloor 4T\rfloor$. For $a\in\mathbb{R}$, we recall that

$$\varkappa_{\lambda,\pi}=\varkappa^{2K}_{\lambda,\pi}=\frac{2K\mathbf{n}_\lambda}{\mathbf{a}_\lambda\pi}+\varepsilon_\lambda$$

and

$$J_a^\lambda \coloneqq [\![\lfloor a\mathbf{n}_\lambda\rfloor\,,\lfloor (a+1)\mathbf{n}_\lambda\rfloor -1]\!]$$

and introduce

$$J_{a,K}^\lambda \coloneqq [\![\lfloor (a-3K)\mathbf{n}_\lambda\rfloor\,,\lfloor (a+3K+1)\mathbf{n}_\lambda\rfloor -1]\!].$$

As usual, for $a\in\mathbb{R}$, we are going to build an event $\Omega_{a,T}^{\lambda,\pi}$ depending only on the Poisson processes $N_t^S(i), N_t^M(i)$ and $N_t^P(i)$ for $t\in[0\,,\mathbf{a}_\lambda T]$ and $i\in J_{a,K}^\lambda$ such that

(i) on $\Omega_{a,T}^{\lambda,\pi}$, there exists $\iota^+\colon[0\,,\mathbf{a}_\lambda T]\mapsto J_{a,K}^\lambda$ (resp. $\iota^-\colon[0\,,\mathbf{a}_\lambda T]\mapsto J_{a,K}^\lambda$), non decreasing (resp. non increasing), such that $\eta_t^{\lambda,\pi}(\iota_t^+)=0$ (resp. $\eta_t^{\lambda,\pi}(\iota_t^-)=0$) for all $t\in[0\,,\mathbf{a}_\lambda T]$,

(ii) there exists $q_T>0$ such that for all $a\in\mathbb{R}$, we have $\mathbb{P}\left[\Omega_{a,T}^{\lambda,\pi}\right]\geq q_T$ for all (λ,π) sufficiently close to the regime $\mathcal{R}(0)$.

It is then routine to conclude the proof.

We now fix $\alpha=0.001$ and assume that (λ,π) is sufficienly close to the regime $\mathcal{R}(0)$ in such a way that $\varkappa_{\lambda,\pi}\leq\alpha$.

Step 1. Here we show that for all $b\in\mathbb{R}$, there exists an event $\Omega_{b,0}^{\lambda,\pi}$, depending only on $(N_s^S(i),N_s^M(i),N_s^P(i))_{s\in[0,3\mathbf{a}_\lambda/4],i\in J_b^\lambda}$ such that

(i) on $\Omega_{b,0}^{\lambda,\pi}$, a.s., there is $i\in J_b^\lambda$ such that $\eta_{\mathbf{a}_\lambda s}^{\lambda,\pi}(i)=0$ for all $s\in[0\,,3/4]$;

(ii) $\lim_{\lambda\to 0}\mathbb{P}\left[\Omega_{b,0}^{\lambda,\pi}\right]=1$.

Simply consider the event $\Omega_{b,0}^{\lambda,\pi}=\{\exists i\in J_b^\lambda, N_{3\mathbf{a}_\lambda/4}^S(i)=0\}$. Clearly, point (i) is satisfied, since there is a site in J_b^λ on which no seed falls during $[0\,,3\mathbf{a}_\lambda/4]$. Since $\left|J_b^\lambda\right|\simeq\mathbf{n}_\lambda\simeq 1/(\lambda\log(1/\lambda))$, we deduce that

$$\mathbb{P}\left[\Omega_{b,0}^{\lambda,\pi}\right]=1-(1-e^{-3\mathbf{a}_\lambda/4})^{\mathbf{n}_\lambda}\simeq 1-e^{-1/(\lambda^{1/4}\mathbf{a}_\lambda)}\xrightarrow[\lambda\to 0]{}1,$$

whence (ii).

Step 2. For $\lambda>0$ and $\pi\geq 1$, we put $\mathbf{k}_\lambda\coloneqq\lfloor\lambda^{-3/8}\rfloor$ and observe that $\mathbf{k}_\lambda\ll\mathbf{n}_\lambda$. For $k\in\{1,\dots,K-1\}$, we set

$$\tau_k=\frac{k+1}{4}\text{ and }\tilde\tau_k=\frac{k+1}{4}+\frac{1}{8}.$$

Consider the event $\tilde\Omega_{a,T}^{\lambda,\pi}$ on which points 1, 2 and 3 below are satisfied.

1. The family of Poisson processes $(N_t^M(i))_{t\in[0,\mathbf{a}_\lambda T], i\in J^\lambda_{a,K}}$ has exactly $2(K-1)$ marks in $J^\lambda_{a,K}$, and we call them

$$\{(X_1^\lambda, T_1^\lambda), \dots, (X_{K-1}^\lambda, T_{K-1}^\lambda)\} \text{ and } \{(\tilde{X}_1^\lambda, \tilde{T}_1^\lambda), \dots, (\tilde{X}_{K-1}^\lambda, \tilde{T}_{K-1}^\lambda)\},$$

in such a way that, for all $k \in \{1, \dots, K-1\}$,

$$(X_k^\lambda, T_k^\lambda) \in [\![\lfloor (a-K+k+\frac{1}{3})\mathbf{n}_\lambda\rfloor \,,\, \lfloor (a-K+k+\frac{2}{3})\mathbf{n}_\lambda\rfloor]\!] \times [(\tau_k - 1/12)\mathbf{a}_\lambda \,,\, (\tau_k - \varkappa_{\lambda,\pi})\,\mathbf{a}_\lambda]$$

and

$$(\tilde{X}_k^\lambda, \tilde{T}_k^\lambda) \in [\![\lfloor (a+K-(k+1)+\frac{1}{3})\mathbf{n}_\lambda\rfloor \,,\, \lfloor (a+K-(k+1)+\frac{2}{3})\mathbf{n}_\lambda\rfloor]\!] \\ \times [(\tilde{\tau}_k - 1/12)\mathbf{a}_\lambda \,,\, (\tilde{\tau}_k - \varkappa_{\lambda,\pi})\,\mathbf{a}_\lambda].$$

(See Figure II.4 for a graphical example.)

2. The family of Poisson processes $(N_t^S(i))_{t\geq 0, i\in J^\lambda_{a,K}}$ satisfies, for all $k \in \{1, \dots, K-1\}$,
   a) there are $j_g \in [\![\lfloor (a-K+k)\mathbf{n}_\lambda\rfloor \,,\, \lfloor (a-K+k+1/4)\mathbf{n}_\lambda\rfloor]\!]$ and $j_d \in [\![\lfloor (a-K+k+3/4)\mathbf{n}_\lambda\rfloor \,,\, \lfloor (a-K+k+1)\mathbf{n}_\lambda - 1\rfloor]\!]$ such that
   $$N^S_{\mathbf{a}_\lambda(\tau_k+1/4)}(j_g) - N^S_{\mathbf{a}_\lambda(\tau_k-1/2)}(j_g) = N^S_{\mathbf{a}_\lambda(\tau_k+1/4)}(j_d) - N^S_{\mathbf{a}_\lambda(\tau_k-1/2)}(j_d) = 0;$$
   b) for all $i \in [\![X_k^\lambda - \mathbf{k}_\lambda \,,\, X_k^\lambda + \mathbf{k}_\lambda]\!]$,
   $$N^S_{\mathbf{a}_\lambda(\tau_k-1/12)}(i) - N^S_{\mathbf{a}_\lambda(\tau_k-1/2)}(i) > 0;$$
   c) there is $j_0 \in [\![X_k^\lambda - \mathbf{k}_\lambda \,,\, X_k^\lambda + \mathbf{k}_\lambda]\!]$ such that
   $$N^S_{\mathbf{a}_\lambda(\tau_k+1/4)}(j_0) - N^S_{\mathbf{a}_\lambda(\tau_k-1/12)}(j_0) = 0.$$

3. The family of Poisson processes $(N_t^S(i))_{t\geq, i\in J^\lambda_{a,K}}$ satisfies, for all $k \in \{1, \dots, K-1\}$,
   a) there are $j_g \in [\![\lfloor (a+K-(k+1))\mathbf{n}_\lambda\rfloor \,,\, \lfloor (a+K-(k+1)+1/4)\mathbf{n}_\lambda\rfloor]\!]$ and $j_d \in [\![\lfloor (a+K-(k+1)+3/4)\mathbf{n}_\lambda\rfloor \,,\, \lfloor (a+K-(k+1)+1)\mathbf{n}_\lambda - 1\rfloor]\!]$ such that
   $$N^S_{\mathbf{a}_\lambda(\tilde{\tau}_k+1/4)}(j_g) - N^S_{\mathbf{a}_\lambda(\tilde{\tau}_k-1/2)}(j_g) = N^S_{\mathbf{a}_\lambda(\tilde{\tau}_k+1/4)}(j_d) - N^S_{\mathbf{a}_\lambda(\tilde{\tau}_k-1/2)}(j_d) = 0;$$
   b) for all $i \in [\![\tilde{X}_k^\lambda - \mathbf{k}_\lambda \,,\, \tilde{X}_k^\lambda + \mathbf{k}_\lambda]\!]$,
   $$N^S_{\mathbf{a}_\lambda(\tilde{\tau}_k-1/12)}(i) - N^S_{\mathbf{a}_\lambda(\tilde{\tau}_k-1/2)}(i) > 0;$$
   c) there is $j_0 \in [\![\tilde{X}_k^\lambda - \mathbf{k}_\lambda \,,\, \tilde{X}_k^\lambda + \mathbf{k}_\lambda]\!]$ such that
   $$N^S_{\mathbf{a}_\lambda(\tilde{\tau}_k+1/4)}(j_0) - N^S_{\mathbf{a}_\lambda(\tilde{\tau}_k-1/12)}(j_0) = 0.$$

We also introduce the event

$$\Omega^{P,K}_{\lambda,\pi} = \left(\bigcap_{k=1}^{K-1} \Omega^{P,2K,2K}_{\lambda,\pi}\left(\frac{X^\lambda_k}{\mathbf{n}_\lambda}, \frac{T^\lambda_k}{\mathbf{a}_\lambda}\right)\right) \cap \left(\bigcap_{k=1}^{K-1} \Omega^{P,2K,2K}_{\lambda,\pi}\left(\frac{\tilde{X}^\lambda_k}{\mathbf{n}_\lambda}, \frac{\tilde{T}^\lambda_k}{\mathbf{a}_\lambda}\right)\right),$$

recall Definition II.4.7.

Finally, we set

$$\Omega^{\lambda,\pi}_{a,T} = \tilde{\Omega}^{\lambda,\pi}_{a,T} \cap \Omega^{P,K}_{\lambda,\pi} \cap \Omega^{\lambda,\pi}_{a-K,0} \cap \Omega^{\lambda,\pi}_{a+K-1,0}.$$

Step 3. Here we prove (ii).

The probability of the event on which N^M satisfies 1. does not depend on $a \in \mathbb{R}$ by invariance by spatial translation. We also can construct N^M using a Poisson measure π_M on $\mathbb{R} \times [0\,,\infty)$ with intensity measure $\mathrm{d}x\,\mathrm{d}t$, independent of N^S and N^P, by setting, for all $i \in \mathbb{Z}$

$$N^M_t(i) = \pi_M(i_\lambda \times [0\,, t/\mathbf{a}_\lambda]).$$

As usual, for all $\lambda > 0$ small enough, the probability of the event on which N^M satisfies 1 is then bounded from below by some constant $2q_T > 0$, which does not depend on $a \in \mathbb{R}$ nor on $\lambda > 0$ and $\pi \geq 1$. We write $\mathbb{P}_M$ for the conditional probability w.r.t. π_M.

Let now $k \in \{1, \dots, K-1\}$. The probability of 2-(a) tends to 1. Indeed, treating e.g. the case of j_g, there holds, recalling $\mathbf{n}_\lambda \simeq 1/(\lambda \mathbf{a}_\lambda)$ and $\mathbf{a}_\lambda = \log(1/\lambda)$,

$$\mathbb{P}\left[\exists j \in [\![\lfloor (a-K+k)\mathbf{n}_\lambda \rfloor\,, \lfloor (a-K+k+1/4)\mathbf{n}_\lambda \rfloor]\!], N^S_{\mathbf{a}_\lambda(\tau_k+1/4)}(j) - N^S_{\mathbf{a}_\lambda(\tau_k-1/2)}(j) = 0\right]$$
$$= 1 - (1 - e^{-(3/4)\mathbf{a}_\lambda})^{\mathbf{n}_\lambda/4} \simeq 1 - e^{-\mathbf{n}_\lambda \lambda^{3/4}/4} \xrightarrow[\lambda\to 0]{} 1.$$

The probability of 2-(b) (conditionally on π_M) also tends to 1. Indeed, it equals

$$(1 - e^{-5\mathbf{a}_\lambda/12})^{2\mathbf{k}_\lambda+1} \simeq e^{-2\mathbf{k}_\lambda \lambda^{5/12}} \xrightarrow[\lambda\to 0]{} 1$$

since $\mathbf{k}_\lambda = \lfloor \lambda^{-3/8} \rfloor$ and since $3/8 < 5/12$. Finally, the probability of 2-(c) (conditionally on π_M) also tends to 1, since it equals

$$1 - (1 - e^{-\mathbf{a}_\lambda/3})^{2\mathbf{k}_\lambda+1} \simeq 1 - e^{-2\mathbf{k}_\lambda \lambda^{1/3}}$$

which tends to 1 when $\lambda \to 0$, since $1/3 < 3/8$.

Similar considerations hold for Point 3.

Finally, since π_M is independent of the processes family $(N^S_t(i))_{t\geq 0, i\in\mathbb{Z}}$ and $(N^P_t(i))_{t\geq 0, i\in\mathbb{Z}}$, Lemma II.4.3 directly implies that, using space/time stationarity, for all $k \in \{1, \dots, K-1\}$,

$$\mathbb{P}_M\left[\Omega^{P,2K,2K}_{\lambda,\pi}(X^\lambda_k/\mathbf{n}_\lambda, T^\lambda_k/\mathbf{a}_\lambda)\right] = \mathbb{P}_M\left[\Omega^{P,2K,2K}_{\lambda,\pi}(\tilde{X}^\lambda_k/\mathbf{n}_\lambda, \tilde{T}^\lambda_k/\mathbf{a}_\lambda)\right]$$

tends to 1 when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(0)$.

All this implies that there exists $q_T > 0$ such that $\mathbb{P}\left[\Omega^{\lambda,\pi}_{a,T}\right] > q_T$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(0)$.

Step 4. Here we work on $\Omega_{a,T}^{\lambda,\pi}$ and we prove that, for all $k \in \{1, \dots, K-1\}$, if there is no burning tree in J_{a-K+k}^{λ} at time $(\tau_k - 1/2)\mathbf{a}_\lambda$, then there is $i \in J_{a-K+k}^{\lambda}$ such that $\eta_{\mathbf{a}_\lambda t}^{\lambda,\pi}(i) = 0$ for all $t \in [\tau_k\,, \tau_k + 1/4]$. We distinguish two cases.

- If the zone $[\![X_k^\lambda - \mathbf{k}_\lambda\,, X_k^\lambda + \mathbf{k}_\lambda]\!]$ is completely occupied at time $T_k^\lambda-$, then each site burns at least one time (i.e. each site in this zone is ignited and then extinguished) during $[T_k^\lambda\,, T_k^\lambda + \mathbf{a}_\lambda \varkappa_{\lambda,\pi}]$, thanks to $\Omega_{\lambda,\pi}^{P,2K,2K}(X_k^\lambda/\mathbf{n}_\lambda, T_k^\lambda/\mathbf{a}_\lambda)$, recall **Macro**(0) in Subsection II.4.4. Since no seed falls on j_0, which belongs to this zone, during

  $$[\mathbf{a}_\lambda(\tau_k - 1/12)\,, \mathbf{a}_\lambda(\tau_k + 1/4)] \supset [T_k^\lambda + \mathbf{a}_\lambda \varkappa_{\lambda,\pi}\,, \mathbf{a}_\lambda(\tau_k + 1/4)] \supset [\mathbf{a}_\lambda \tau_k\,, \mathbf{a}_\lambda(\tau_k + 1/4)],$$

  we deduce that $\eta_{\mathbf{a}_\lambda s}^{\lambda,\pi}(j_0) = 0$ for all $s \in [\tau_k\,, \tau_k + 1/4]$.

- Assume now that there exists $i_0 \in [\![X_k^\lambda - \mathbf{k}_\lambda\,, X_k^\lambda + \mathbf{k}_\lambda]\!]$ that is vacant at time $T_k^\lambda-$. Recall that there is no match falling in J_a^λ during $[\mathbf{a}_\lambda(\tau_k - 1/2)\,, T_k^\lambda)$, that on each site of $[\![X_k^\lambda - \mathbf{k}_\lambda\,, X_k^\lambda + \mathbf{k}_\lambda]\!]$, at least one seed falls during $[\mathbf{a}_\lambda(\tau_k - 1/2)\,, \mathbf{a}_\lambda(\tau_k - 1/12)] \subset [\mathbf{a}_\lambda(\tau_k - 1/2)\,, T_k^\lambda)$ and that there is no burning tree in J_{a-K+k}^λ at time $\mathbf{a}_\lambda(\tau_k - 1/2)$. Then necessarily, a fire starting at some $i'_M \notin J_{a-K+k}^\lambda$ at some time $t'_M < T_k^\lambda$, has made vacant i_0. Assume e.g. that $i'_M < \lfloor (a-K+k)\mathbf{n}_\lambda \rfloor$ and observe that $i'_M < j_g < i_0$. The fire (i'_M, t'_M) has then also necessarily made vacant j_g during $(\mathbf{a}_\lambda(\tau_k - 1/2)\,, T_k^\lambda)$. Since no seed falls on j_g during $[\mathbf{a}_\lambda(\tau_k - 1/2)\,, \mathbf{a}_\lambda(\tau_k + 1/4)]$, we deduce that j_g remains vacant during $[\mathbf{a}_\lambda \tau_k\,, \mathbf{a}_\lambda(\tau_k + 1/4)]$.

Step 5. We can show, exactly as above, that, on $\Omega_{a,T}^{\lambda,\pi}$, if there is no burning tree in $J_{a+K-(k+1)}^\lambda$ at time $(\tilde{\tau}_k - 1/2)\mathbf{a}_\lambda$, for some $k \in \{1, \dots, K-1\}$, then there is $i \in J_{a+K-(k+1)}^\lambda$ such that $\eta_{\mathbf{a}_\lambda t}^{\lambda,\pi}(i) = 0$ for all $t \in [\tilde{\tau}_k\,, \tilde{\tau}_k + 1/4]$.

Step 6. To conclude the proof, we now prove by induction (see Figure II.4) that for all $k \in \{1, \dots, K-1\}$

▷ there exists $i_k \in J_{a-K+k}^\lambda$ such that $\eta_{\mathbf{a}_\lambda t}^{\lambda,\pi}(i_k) = 0$ for all $t \in [\tau_k\,, \tau_k + 1/4]$;

▷ there exists $j_k \in J_{a+K-(k+1)}^\lambda$ such that $\eta_{\mathbf{a}_\lambda t}^{\lambda,\pi}(j_k) = 0$ for all $t \in [\tilde{\tau}_k\,, \tilde{\tau}_k + 1/4]$;

▷ there is no burning tree in $[\![i_k\,, j_k]\!]$ at time $\mathbf{a}_\lambda \tau_k$ nor at time $\mathbf{a}_\lambda \tilde{\tau}_k$.

• At time 0, all sites are vacant. Thus, there are $i_0 \in J_{a-K}^\lambda$ and $j_0 \in J_{a+K-1}^\lambda$ which remain vacant until time $3\mathbf{a}_\lambda/4$ (thanks to $\Omega_{a-K,0}^{\lambda,\pi} \cap \Omega_{a+K-1,0}^{\lambda,\pi}$). Since no match falls in $[\![i_0\,, j_0]\!]$ until time $T_1^\lambda \geq \mathbf{a}_\lambda(1/2 - 1/12) = 5\mathbf{a}_\lambda/12$, there is no burning tree at all in $[\![i_0\,, j_0]\!]$ during $[0\,, 5\mathbf{a}_\lambda/12)$ (no match falling outside $[\![i_0\,, j_0]\!]$ during $[0\,, 5\mathbf{a}_\lambda/12)$ can affect this zone).

Thus, Step 4 shows that there are $i_1 \in J_{a-K+1}^\lambda$ which is vacant during $[\mathbf{a}_\lambda/2\,, 3\mathbf{a}_\lambda/4]$ (because $\tau_1 - 1/2 = 0$) and $i_2 \in J_{a-K+2}^\lambda$ which is vacant during $[3\mathbf{a}_\lambda/4\,, \mathbf{a}_\lambda]$ (because $\tau_2 - 1/2 = 1/4 < 5/12$). Similarly, Step 5 above shows that there are $j_1 \in J_{a+K-2}^\lambda$ which is vacant during $[5\mathbf{a}_\lambda/8\,,7\mathbf{a}_\lambda/8]$ (because $\tilde{\tau}_1-1/2=1/8<5/12$) and $j_2\in J^\lambda_{a+K-3}$ which is vacant during $[7\mathbf{a}_\lambda/8\,,9\mathbf{a}_\lambda/8]$ (because $\tilde{\tau}_2-1/2=3/8<5/12$).

Since $T^\lambda_1\le(1/2-\varkappa_{\lambda,\pi})\mathbf{a}_\lambda$ and $|X^\lambda_1-i_0|\le|X^\lambda_1-j_0|\le 2K\mathbf{n}_\lambda$, as seen in **Macro**(0) in Subsection II.4.4 (recall that we work on $\Omega^{P,2K,2K}_{\lambda,\pi}(X^\lambda_1/\mathbf{n}_\lambda,T^\lambda_1/\mathbf{a}_\lambda)$), there is no more burning tree in $[\![i_0\,,j_0]\!]$ at time $T^\lambda_1+\mathbf{a}_\lambda\varkappa_{\lambda,\pi}\le\mathbf{a}_\lambda/2=\mathbf{a}_\lambda\tau_1$. Since no other match falls in $[\![i_0\,,j_0]\!]$ during $[T^\lambda_1+\mathbf{a}_\lambda\varkappa_{\lambda,\pi}\,,\mathbf{a}_\lambda/2]$, we deduce that there is also no burning tree in $[\![i_0\,,j_0]\!]\supset[\![i_1\,,j_1]\!]$ at time $\mathbf{a}_\lambda\tau_1$ (because i_0 and j_0 remain vacant until $\mathbf{a}_\lambda/2$).

Since no match falls in $[\![i_1\,,j_0]\!]$ during $[\mathbf{a}_\lambda\tau_1\,,\tilde{T}^\lambda_1)$, we deduce that there is no burning tree in $[\![i_1\,,j_0]\!]$ at time $\tilde{T}^\lambda_1-$. But i_1 and j_0 remain vacants during $[\tilde{T}^\lambda_1\,,\tilde{T}^\lambda_1+\mathbf{a}_\lambda\varkappa_{\lambda,\pi}]\subset[\mathbf{a}_\lambda\tau_1\,,\mathbf{a}_\lambda\tilde{\tau}_1]$ and only one match falls in $[\![i_1\,,j_0]\!]$ during $[\tilde{T}^\lambda_1\,,\tilde{T}^\lambda_1+\mathbf{a}_\lambda\varkappa_{\lambda,\pi}]$. Hence, recall **Macro**(0) in Subsection II.4.4, there is no more burning tree in $[\![i_1\,,j_0]\!]$ at time $\tilde{T}^\lambda_1+\mathbf{a}_\lambda\varkappa_{\lambda,\pi}$. We easily deduce that there is also no burning tree in $[\![i_1\,,j_1]\!]\subset[\![i_1\,,j_0]\!]$ at time $\mathbf{a}_\lambda\tilde{\tau}_1$.

Similarly, since $i_0<i_1<i_2<j_2<j_1<j_0$ and thanks to $\Omega^{P,K}_{\lambda,\pi}$, there is no more burning tree in $[\![i_1\,,j_1]\!]\supset[\![i_2\,,j_2]\!]$ at time τ_2 nor in $[\![i_2\,,j_1]\!]\supset[\![i_2\,,j_2]\!]$ at time $\tilde{\tau}_2$.

- Assume now that there is $k\in\{2,\dots,K-2\}$ such that, for all $l\le k$,

  ▷ there exists $i_l\in J^\lambda_{a-K+l}$ such that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}(i_l)=0$ for all $t\in[\mathbf{a}_\lambda\tau_l\,,\mathbf{a}_\lambda(\tau_l+1/4)]$;

  ▷ there exists $j_l\in J^\lambda_{a+K-(l+1)}$ such that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}(j_l)=0$ for all $t\in[\mathbf{a}_\lambda\tilde{\tau}_l\,,\mathbf{a}_\lambda(\tilde{\tau}_l+1/4)]$;

  ▷ there is no burning tree in $[\![i_l\,,j_l]\!]$ at time $\mathbf{a}_\lambda\tau_l$ nor at time $\mathbf{a}_\lambda\tilde{\tau}_l$.

Since there is no burning tree in $J^\lambda_{a-K+k+1}\subset[\![i_{k-1}\,,j_{k-1}]\!]$ at time $\mathbf{a}_\lambda\tau_{k-1}=\mathbf{a}_\lambda(\tau_{k+1}-1/2)$, see Step 4, there is $i_{k+1}\in J^\lambda_{a-K+k+1}$ which is vacant during $[\mathbf{a}_\lambda\tau_{k+1}\,,\mathbf{a}_\lambda(\tau_{k+1}+1/4)]$. Furthermore, observe that i_k and j_k remain vacants during $[\mathbf{a}_\lambda\tilde{\tau}_k\,,\mathbf{a}_\lambda\tau_{k+1}]$, no match falls in $[\![i_k\,,j_k]\!]$ during $[\mathbf{a}_\lambda\tilde{\tau}_k\,,T^\lambda_{k+1})\subset[\mathbf{a}_\lambda\tilde{\tau}_k\,,\mathbf{a}_\lambda(\tau_{k+1}-\varkappa_{\lambda,\pi})]\subset[\mathbf{a}_\lambda\tilde{\tau}_k\,,\mathbf{a}_\lambda\tau_{k+1}]$ and there is no burning tree in $[\![i_k\,,j_k]\!]$ at time $\mathbf{a}_\lambda\tilde{\tau}_k$. Thus, as seen in **Macro**(0) in Subsection II.4.4, on $\Omega^{P,2K,2K}_{\lambda,\pi}\left(X^\lambda_{k+1}/\mathbf{n}_\lambda,T^\lambda_{k+1}/\mathbf{a}_\lambda\right)$, there is no more burning tree in $[\![i_k\,,j_k]\!]$ at time $T^\lambda_{k+1}+\mathbf{a}_\lambda\varkappa_{\lambda,\pi}$, nor at time $\mathbf{a}_\lambda\tau_{k+1}$.

Since there is no burning tree in $J^\lambda_{a+K-(k+2)}\subset[\![i_{k-1}\,,j_{k-1}]\!]$ at time $\mathbf{a}_\lambda\tilde{\tau}_{k-1}=\mathbf{a}_\lambda(\tilde{\tau}_{k+1}-1/2)$, we deduce by Step 5 that there is $j_{k+1}\in J^\lambda_{a+K-(k+2)}$ which is vacant during $[\mathbf{a}_\lambda\tilde{\tau}_{k+1}\,,\mathbf{a}_\lambda(\tilde{\tau}_{k+1}+1/4)]$. Furthermore, observe that i_{k+1} and j_k remain vacants during $[\mathbf{a}_\lambda\tau_{k+1}\,,\mathbf{a}_\lambda\tilde{\tau}_{k+1}]$, no match falls in $[\![i_{k+1}\,,j_k]\!]$ during $[\mathbf{a}_\lambda\tau_{k+1}\,,\tilde{T}^\lambda_{k+1})\subset[\mathbf{a}_\lambda\tau_{k+1}\,,\mathbf{a}_\lambda(\tilde{\tau}_{k+1}-\varkappa_{\lambda,\pi})]$ and there is no burning tree in $[\![i_{k+1}\,,j_k]\!]$ at time $\mathbf{a}_\lambda\tau_{k+1}$. Thus, as seen in **Macro**(0) in Subsection II.4.4, on $\Omega^{P,2K,2K}_{\lambda,\pi}\left(\frac{\tilde{X}^\lambda_{k+1}}{\mathbf{n}_\lambda},\frac{\tilde{T}^\lambda_{k+1}}{\mathbf{a}_\lambda}\right)$, there is no more burning tree in $[\![i_{k+1}\,,j_k]\!]$ at time $\tilde{T}^\lambda_{k+1}+\mathbf{a}_\lambda\varkappa_{\lambda,\pi}$ nor at time $\mathbf{a}_\lambda\tilde{\tau}_{k+1}$, as usual.

- By the induction above, we deduce that there are

$$\iota^+:[0\,,T]\to J^\lambda_{a,K}$$

non decreasing, such that for all $t \in [0\,,T]$, $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}(\iota^+_{\mathbf{a}_\lambda t}) = 0$ and

$$\iota^- : [0\,,T] \to J^\lambda_{a,K}$$

non increasing, such that for all $t \in [0\,,T]$, $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}(\iota^-_{\mathbf{a}_\lambda t}) = 0$. This together with Step 3 conclude the proof in the regime $\mathcal{R}(0)$. □

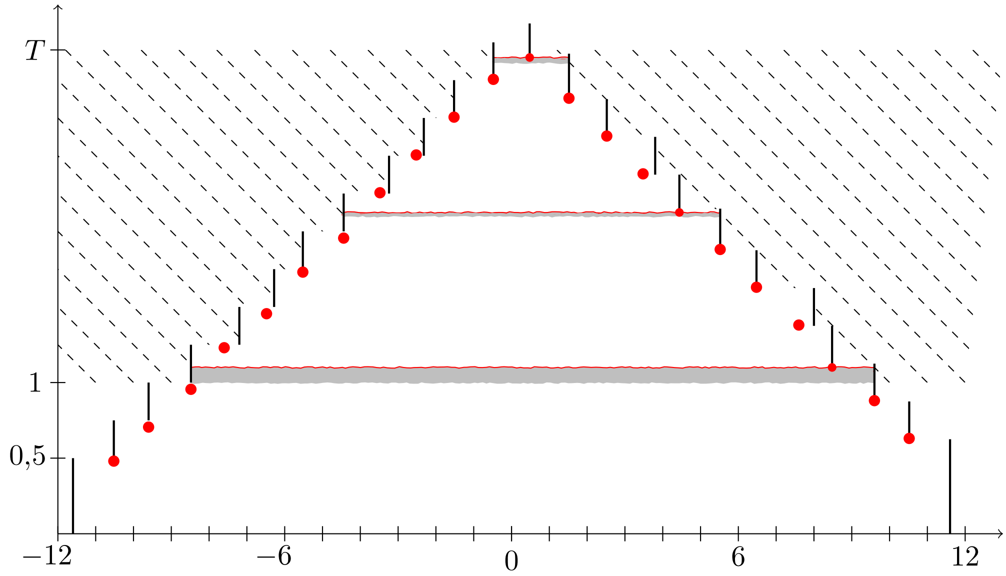

Figure II.4.: The sweet event

Here $T = 3.2, K = 12$ and $a \in [0\,,1)$. The marks of π_M (matches) are represented as $\bullet$'s. The filled zones represent macroscopic zones ($Z^{\lambda,\pi}_{\mathbf{a}_\lambda t}(x) = 1$). In the rest of the space, we always have $Z^{\lambda,\pi}_{\mathbf{a}_\lambda t}(x) < 1$. The plain vertical segments represent vacants sites i.e. sites where no seed falls after being propagated. Remark that sometimes the vacant site is above the match (that is in an interval with length $2\mathbf{k}_\lambda$) and sometimes it is next to the match (that is an i^g or an i^d).

II.6. Localization of the result

In this section, we localize Theorems II.2.4 and II.2.10.

II.6.1. Localization in the regime $\mathcal{R}(p)$

The following Theorem will be proved in Section II.8 in the case $p > 0$ and in Section II.9 in the case $p = 0$.

Theorem II.6.1. *Let $A > 0$ and $p \geq 0$ be fixed. Consider for each $\lambda \in (0\,,1]$ and each $\pi \geq 1$, the process $(Z_t^{\lambda,\pi,A}(x), D_t^{\lambda,\pi,A})_{t\geq 0,x\in\mathbb{R}}$ associated with the $(\lambda,\pi,A)-FFP$. Consider also the $A-LFFP(p)$ $(Z_t^A(x), H_t^A(x), F_t^A(x))_{t\geq 0,x\in\mathbb{R}}$ and the associated $(D_t^A(x))_{t\geq 0,x\in\mathbb{R}}$. We assume that $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$.*

1. *For any $T > 0$, any finite subset $\{x_1, \ldots, x_q\} \subset \mathbb{R}$,*

$$(Z_t^{\lambda,\pi,A}(x_i), D_t^{\lambda,\pi,A}(x_i))_{t\in[0,T],i=1,\ldots,q}$$

   goes in law to $(Z_t^A(x_i), D_t^A(x_i))_{t\in[0,T],i=1,\ldots,q}$ in $\mathbb{D}([0\,,T],\mathbb{R}\times(\mathcal{I}\cup\{\emptyset\}))$. Here $\mathbb{D}([0\,,T],\mathbb{R}\times(\mathcal{I}\cup\{\emptyset\}))$ is endowed with the distance $\mathbf{d}_T$.

2. *For any subset $\{(x_1,t_1),\ldots,(x_q,t_q)\} \subset \mathbb{R}\times[0\,,\infty)$, $(Z_{t_i}^{\lambda,\pi,A}(x_i), D_{t_i}^{\lambda,\pi,A}(x_i))_{i=1,\ldots,q}$ goes in law to $(Z_{t_i}^A(x_i), D_{t_i}^A(x_i))_{i=1,\ldots,q}$ in $(\mathbb{R}\times(\mathcal{I}\cup\{\emptyset\}))^q$. Here $\mathcal{I}\cup\{\emptyset\}$ is endowed with $\boldsymbol{\delta}$.*

3. *For all $t > 0$,*

$$\left(\frac{\log(|C(\eta_{\mathbf{a}_\lambda t}^{\lambda,\pi,A},0)|)}{\log(1/\lambda)}\boldsymbol{1}_{\{|C(\eta_{\mathbf{a}_\lambda t}^{\lambda,\pi,A},0)|\geq 1\}}\right)\wedge 1$$

   goes in law to $Z_t^A(0)$.

Assuming for a moment that this theorem holds true, we conclude the proof of Theorem II.2.4.

Proof of Theorem II.2.4. Let us first prove 1. Consider a continuous bounded function $\Psi : \mathbb{D}([0,T],\mathbb{R}\times(\mathcal{I}\cup\{\emptyset\}))^q \mapsto \mathbb{R}$. We have to prove that $G_{\lambda,\pi}(\Psi)$ tends to 0 when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$, where

$$G_{\lambda,\pi}(\Psi) = \mathbb{E}\left[\Psi\left((Z_t^{\lambda,\pi}(x_i), D_t^{\lambda,\pi}(x_i))_{t\in[0,T],i=1,\ldots,q}\right)\right] - \mathbb{E}\left[\Psi\left((Z_t(x_i), D_t(x_i))_{t\in[0,T],i=1,\ldots,q}\right)\right].$$

Using now Propositions II.3.5 and II.5.2, we observe that for any $A > 2\max_{i=1,\ldots,q}|x_i|$, there holds that, for all (λ,π) sufficiently close to the regime $\mathcal{R}(p)$,

$$
\begin{aligned}
&|G_{\lambda,\pi}(\Psi)| \\
\leq &2\|\Psi\|_\infty \mathbb{P}\left[(Z_t^{\lambda,\pi,A}(x), D_t^{\lambda,\pi,A}(x))_{t\in[0,T],x\in[-A/2,A/2]} \neq (Z_t^{\lambda,\pi}(x), D_t^{\lambda,\pi}(x))_{t\in[0,T],x\in[-A/2,A/2]}\right] \\
&+ 2\|\Psi\|_\infty \mathbb{P}\left[(Z_t^{A}(x), D_t^{A}(x))_{t\in[0,T],x\in[-A/2,A/2]} \neq (Z_t(x), D_t(x))_{t\in[0,T],x\in[-A/2,A/2]}\right] \\
&+ \left|\mathbb{E}\left[\Psi\left((Z_t^{\lambda,\pi,A}(x_i), D_t^{\lambda,\pi,A}(x_i))_{t\in[0,T],i=1,\dots,q}\right)\right] - \mathbb{E}\left[\Psi\left((Z_t^{A}(x_i), D_t^{A}(x_i))_{t\in[0,T],i=1,\dots,q}\right)\right]\right| \\
\leq &4\|\Psi\|_\infty C_T e^{-\alpha_T A} \\
&+ \left|\mathbb{E}\left[\Psi\left((Z_t^{\lambda,\pi,A}(x_i), D_t^{\lambda,\pi,A}(x_i))_{t\in[0,T],i=1,\dots,q}\right)\right] - \mathbb{E}\left[\Psi\left((Z_t^{A}(x_i), D_t^{A}(x_i))_{t\in[0,T],i=1,\dots,q}\right)\right]\right|.
\end{aligned}
$$

Thus Proposition II.6.1-1 implies that

$$|G_{\lambda,\pi}(\Psi)| \leq 5\|\Psi\|_\infty C_T e^{-\alpha_T A},$$

for all (λ,π) sufficiently close to the regime $\mathcal{R}(p)$. We conclude by making A tend to infinity.

Point 2 is checked similarly. The proof of 3 is also similar, since $D_t^{\lambda,\pi}(0) = D_t^{\lambda,\pi,A}(0)$ implies that $C(\eta_{\mathbf{a}_\lambda t}^{\lambda,\pi},0) = C_A(\eta_{\mathbf{a}_\lambda t}^{\lambda,\pi,A},0)$. □

II.6.2. Localization in the regime $\mathcal{R}(\infty, z_0)$

The following Theorem will be proved in the next Section.

Theorem II.6.2. *Let $z_0 \in [0\,,1]$ and $A>0$. Consider for each $\lambda \in (0\,,1]$ and each $\pi \geq 1$ the process $(D_t^{\lambda,\pi,A}(x))_{t\geq 0,x\in\mathbb{R}}$ associated with the $(\lambda,\pi,A)-$FFP. Consider also the LFFP(∞,z_0) $(Y_t(x))_{t\geq0,x\in\mathbb{R}}$ and the associated $(D_t^A(x))_{t\geq0,x\in\mathbb{R}}$ process. We assume that $\lambda\to0$ and $\pi\to\infty$ in the slow regime $\mathcal{R}(\infty,z_0)$.*

1. *For any $T>0$, any finite subset $\{x_1,\dots,x_q\}\subset\mathbb{R}$, $(D_t^{\lambda,\pi,A}(x_i))_{t\in[0,T],i=1,\dots,q}$ goes in law to $(D_t^A(x_i))_{t\in[0,T],i=1,\dots,q}$ in $\mathbb{D}([0\,,T],\mathcal{I})^q$. Here $\mathbb{D}([0\,,T],\mathcal{I})^q$ is endowed with $\boldsymbol{\delta}_T$.*

2. *For any finite subset $\{(x_1,t_1),\dots,(x_q,t_q)\}\subset\mathbb{R}\times[0\,,\infty)$, $(D_{t_i}^{\lambda,\pi,A}(x_i))_{i=1,\dots,q}$ goes in law to $(D_{t_i}^A(x_i))_{t\in[0,T],i=1,\dots,q}$ in $\mathcal{I}^q$, $\mathcal{I}$ being endowed with $\boldsymbol{\delta}$.*

Proof of Theorem II.2.10. The proof easily follows from Proposition II.3.1, Proposition II.5.2 and Theorem II.6.2, as in the proof above. □

II.7. Convergence in the slow regime

The aim of this section is to prove Theorem II.6.2. We thus fix the parameters $A > 0$ and $T > 0$.

We recall that $\mathbf{a}_\lambda = \log(1/\lambda)$, $\mathbf{n}_\lambda = \lfloor 1/(\lambda \mathbf{a}_\lambda) \rfloor$, $\mathbf{m}_\lambda = \lfloor 1/(\lambda \mathbf{a}_\lambda^2) \rfloor$, $\varepsilon_\lambda = 1/\mathbf{a}_\lambda^3$ and that

$$A_\lambda = \lfloor A\mathbf{n}_\lambda \rfloor,$$
$$I_A^\lambda = [\![-A_\lambda \, , A_\lambda]\!].$$

For $x \in \mathbb{R}$, we define

$$(x)_\lambda = [\![\lfloor \mathbf{n}_\lambda x \rfloor - \mathbf{m}_\lambda \, , \lfloor \mathbf{n}_\lambda x \rfloor + \mathbf{m}_\lambda]\!].$$

For $\alpha \in (0\,,1)$, we also define

$$\mathbf{m}_\lambda^\alpha = \left\lfloor \frac{\alpha}{\lambda^{\alpha+(1-\alpha)z_0} \mathbf{a}_\lambda} \right\rfloor,$$
$$(x)_\lambda^\alpha = [\![\lfloor \mathbf{n}_\lambda x \rfloor - \mathbf{m}_\lambda^\alpha \, , \lfloor \mathbf{n}_\lambda x \rfloor + \mathbf{m}_\lambda^\alpha]\!].$$

Observe that $\mathbf{m}_\lambda^\alpha \le \lfloor \alpha \mathbf{n}_\lambda \rfloor$ for all $z_0 \in [0\,,1]$.

II.7.1. Occupation of vacant zone

We start with some easy estimates.

Lemma II.7.1. *Consider a family of i.i.d. Poisson processes $(N_t^S(i))_{t\ge 0, i\in\mathbb{Z}}$. Let $0 < z < 1$, $\alpha \in (0\,,1)$ and $a < b$.*

1. *For $t < z$, $\mathbb{P}\left[\forall i \in [\![\lfloor a\lambda^{-z} \rfloor \, , \lfloor b\lambda^{-z} \rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) > 0\right] \xrightarrow[\lambda\to 0]{} 0$.*
2. *For $t > z$, $\mathbb{P}\left[\forall i \in [\![\lfloor a\lambda^{-z} \rfloor \, , \lfloor b\lambda^{-z} \rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) > 0\right] \xrightarrow[\lambda\to 0]{} 1$.*
3. *For $t \ge 1$, $\mathbb{P}\left[\forall i \in [\![\lfloor a\mathbf{n}_\lambda \rfloor \, , \lfloor b\mathbf{n}_\lambda \rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) > 0\right] \xrightarrow[\lambda\to 0]{} 1$.*
4. *For $t < 1$, $\mathbb{P}\left[\forall i \in [\![\lfloor a\mathbf{m}_\lambda \rfloor \, , \lfloor b\mathbf{m}_\lambda \rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) > 0\right] \xrightarrow[\lambda\to 0]{} 0$.*
5. *For $t > z_0 + \alpha$, $\mathbb{P}\left[\forall i \in [\![-\lfloor a\mathbf{m}_\lambda^\alpha \rfloor \, , \lfloor b\mathbf{m}_\lambda^\alpha \rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) > 0\right] \xrightarrow[\lambda\to 0]{} 1$.*

Proof. To check Lemma II.7.1, observe that, for $k_\lambda \xrightarrow[\lambda\to 0]{} \infty$,

$$\mathbb{P}\left[\forall i \in [\![-\lfloor ak_\lambda \rfloor \, , \lfloor bk_\lambda \rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) > 0\right] \simeq (1 - e^{\mathbf{a}_\lambda t})^{(b-a)k_\lambda} \simeq e^{-(b-a)k_\lambda \lambda^t}. \tag{II.7.1}$$

In order to prove 1 and 2, use (II.7.1) with $k_\lambda = \lambda^{-z}$ and observe that

$$k_\lambda \lambda^t = \lambda^{-z}\lambda^t \xrightarrow[\lambda\to 0]{} \begin{cases} \infty & \text{if } t < z, \\ 0 & \text{if } t > z. \end{cases}$$

To prove 3, use (II.7.1) with $k_\lambda = \mathbf{n}_\lambda$ and observe that, if $t \geq 1$, $\mathbf{n}_\lambda \lambda^t \simeq \lambda^{t-1}/\mathbf{a}_\lambda$ tends to 0 when $\lambda \to 0$. In the same way, 4 can be proved using $k_\lambda = \mathbf{m}_\lambda$ and observing that, if $t < 1$, $\mathbf{m}_\lambda \lambda^t \simeq \lambda^{t-1}/\mathbf{a}_\lambda^2$ tends to ∞ when $\lambda \to 0$.

Finally, prove 5 with (II.7.1) and using $k_\lambda = \mathbf{m}_\lambda^\alpha$ and observing that $\mathbf{m}_\lambda^\alpha \lambda^t \simeq \frac{\alpha}{\mathbf{a}_\lambda} \lambda^{t-\alpha-(1-\alpha)z_0}$ tends to 0 when $\lambda \to 0$ as soon as $t - \alpha - (1-\alpha)z_0 > 0$ (in particular, for $t \geq z_0 + \alpha > \alpha + (1-\alpha)z_0$). $\square$

II.7.2. Height of the barrier

We describe here the time needed for a destroyed microscopic cluster to be regenerated. Assume that a match falls in the site 0 at some time $\mathbf{a}_\lambda t_1 \in (0, \mathbf{a}_\lambda z_0)$. As seen in $\mathbf{Micro}(\infty, z_0)$ in Subsection II.4.4, on a suitable event, the $(\lambda, \pi)-$FFP is well understood around 0 during $[\mathbf{a}_\lambda t_1, \mathbf{a}_\lambda(t_1 + \kappa^z_{\lambda,\pi})]$, for some $0 < z < z_0$ (it can be expressed using the sequence $(T^1_i)_{i\in\mathbb{Z}}$). We then denote by $\Theta^{\lambda,\pi}_{t_1}$ the delay needed for the destroyed cluster to be fully regenerated (after rescaling). We show that $\Theta^{\lambda,\pi}_{t_1} \simeq t_1$.

Lemma II.7.2. *Consider two Poisson processes $(N^S_t(i))_{t\geq 0, i\in\mathbb{Z}}$ and $(N^P_t(i))_{t\geq 0, i\in\mathbb{Z}}$ with respective rates 1 and π, all this processes being independent. Let $0 < t_1 < z_0$. We call $(T^1_i)_{i\in\mathbb{Z}}$ the burning times of the propagation process ignited in 0 at time $\mathbf{a}_\lambda t_1$, recall Definition II.4.6.*

Put, for all $t \geq 0$ and $i \in \mathbb{Z}$, $\zeta^{\lambda,\pi}_t(i) = \min(N^S_t(i), 1)$ and define

$$C^P((\zeta^{\lambda,\pi}_t(i))_{t\geq 0, i\in\mathbb{Z}}, (0, t_1)) = [\![i^g, i^d]\!],$$

recall Definition II.4.8.

We define a process $(\zeta^{\lambda,\pi}_{t_1,t}(i))_{t\in[0,T], i\in\mathbb{Z}}$ in the following way (which is inspired by $\mathbf{Micro}(\infty, z_0)$ in Subsection II.4.4): we put, for all $i \in C^P((\zeta^{\lambda,\pi}_t(i))_{t\geq 0, i\in\mathbb{Z}}, (0, t_1))$

$$\zeta^{\lambda,\pi}_{t_1,t}(i) = \min(N^S_{\mathbf{a}_\lambda t}(i), 1) \text{ for } t \in [0, t_1 + (T^1_i/\mathbf{a}_\lambda))$$

and

$$\zeta^{\lambda,\pi}_{t_1,t}(i) = 2 \begin{cases} \text{for } t \in [t_1 + (T^1_i/\mathbf{a}_\lambda), t_1 + (T^1_{i+1}/\mathbf{a}_\lambda)) & \text{if } i \geq 0, \\ \text{for } t \in [t_1 + (T^1_i/\mathbf{a}_\lambda), t_1 + (T^1_{i-1}/\mathbf{a}_\lambda)) & \text{if } i \leq 0 \end{cases}$$

and

$$\zeta^{\lambda,\pi}_{t_1,t}(i) = \begin{cases} \min(N^S_{\mathbf{a}_\lambda(t+t_1)}(i) - N^S_{\mathbf{a}_\lambda t_1 + T^1_{i+1}}(i), 1) & \text{for } t \in [t_1 + (T^1_{i+1}/\mathbf{a}_\lambda), T] \text{ if } i \geq 0, \\ \min(N^S_{\mathbf{a}_\lambda(t+t_1)}(i) - N^S_{\mathbf{a}_\lambda t_1 + T^1_{i-1}}(i), 1) & \text{for } t \in [t_1 + (T^1_{i-1}/\mathbf{a}_\lambda), T] \text{ if } i \leq 0. \end{cases}$$

For all $i \notin C^P((\zeta^{\lambda,\pi}_t(i))_{t\geq 0, i\in\mathbb{Z}}, (0, t_1))$ and all $t \in [0, T]$, we put

$$\zeta^{\lambda,\pi}_{t_1,t}(i) = \min(N^S_{\mathbf{a}_\lambda t}(i), 1).$$

We finally define

$$\Theta^{\lambda,\pi}_{t_1} = \inf\left\{ t > t_1 : \forall i \in C^P((\zeta^{\lambda,\pi}_t(i))_{t\geq 0, i\in\mathbb{Z}}, (0, t_1)), \zeta^{\lambda,\pi}_{t_1,t}(i) = 1 \right\}.$$

Then, for all $\delta > 0$, as $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(\infty, z_0)$, there holds

$$\lim_{\lambda,\pi} \mathbb{P}\left[|\Theta^{\lambda,\pi}_{t_1} - t_1| \geq \delta\right] = 0.$$

The process $(\zeta^{\lambda,\pi}_{t_1,t}(i))_{i\in\mathbb{Z},t\geq 0}$ is closely related to the process observed in $\mathbf{Micro}(\infty, z_0)$ in Subsection II.4.4 (on a suitable event).

Proof. We divide the proof in two steps. We first define a simplest process with an instantaneous propagation: if a match falls in a cluster, it destroys instantaneously the entire connected component. The time needed for a microscopic cluster to become again occupied is almost t_1. Secondly, we flank the killed cluster $C^P((\zeta^{\lambda,\pi}_t(i))_{t\geq 0,i\in\mathbb{Z}}, (0, t_1))$ to estimate the time to become again occupied.

Step 1. Let $0 < \tau_1 < z_0$ be fixed. Put $\vartheta^\lambda_t(i) = \min(N^S_{\mathbf{a}_\lambda t}(i), 1)$ and $\vartheta^\lambda_{\tau_1,t}(i) = \min(N^S_{\mathbf{a}_\lambda(\tau_1+t)}(i) - N^S_{\mathbf{a}_\lambda\tau_1}(i), 1)$ for all $t > 0$ and $i \in \mathbb{Z}$. We define the time needed for the destroyed cluster to be fully regenerated

$$\Xi^\lambda_{\tau_1} = \inf\left\{t > 0 : \forall i \in C(\vartheta^\lambda_{\tau_1}, 0),\ \vartheta^\lambda_{\tau_1,t}(i) = 1\right\}.$$

Then for all $\delta > 0$,

$$\lim_{\lambda\to 0} \mathbb{P}\left[|\Xi^\lambda_{\tau_1} - \tau_1| \geq \delta\right] = 0.$$

Indeed, we write, for $h > 0$,

$$\mathbb{P}\left[\Xi^\lambda_{\tau_1} \leq h\right] = \mathbb{P}\left[N^S_{\mathbf{a}_\lambda\tau_1}(0) = 0\right] + \sum_{k\geq 1}\sum_{j=0}^{k-1} \mathbb{P}\Big[N^S_{\mathbf{a}_\lambda\tau_1}(j-k) = N^S_{\mathbf{a}_\lambda\tau_1}(j+1) = 0,$$
$$\forall i \in [\![j-k+1\,, j]\!], N^S_{\mathbf{a}_\lambda\tau_1}(i) > 0, N^S_{\mathbf{a}_\lambda(\tau_1+h)}(i) > N^S_{\mathbf{a}_\lambda\tau_1}(i)\Big],$$

that is

$$\begin{aligned}
\mathbb{P}\left[\Xi^\lambda_{\tau_1} \leq h\right] &= \lambda^{\tau_1} + \sum_{k\geq 1}\sum_{j=0}^{k-1} \lambda^{\tau_1} \times \lambda^{\tau_1} \times \Big((1-\lambda^{\tau_1})(1-\lambda^h)\Big)^k \\
&= \lambda^{\tau_1} + \lambda^{2\tau_1} \sum_{k\geq 1} k(\Big((1-\lambda^{\tau_1})(1-\lambda^h)\Big)^k \\
&= \lambda^{\tau_1} + \frac{\lambda^{2\tau_1}}{(1-(1-\lambda^{\tau_1})(1-\lambda^h))^2}(1-\lambda^{\tau_1})(1-\lambda^h) \\
&= \lambda^{\tau_1} + \frac{\lambda^{2\tau_1}}{(\lambda^{\tau_1}+\lambda^h-\lambda^{\tau_1+h})^2}(1-\lambda^{\tau_1})(1-\lambda^h).
\end{aligned}$$

This quantity obviously tends to 1 as $\lambda \to 0$ if $h > \tau_1$ and to 0 if $h < \tau_1$.

Step 2. Let $z \in (t_1\,, z_0)$ and define $\Omega^{P,z}_{\lambda,\pi}(0,t_1)$, recall Definition II.4.7. Set

$$\tilde{\Omega}^{P,z}_{\lambda,\pi}(0,t_1) := \Omega^{P,z}_{\lambda,\pi}(0,t_1) \cap \{\exists i_1 \in [\![0\,, \lfloor \lambda^{-z} \rfloor]\!], N^S_{\mathbf{a}_\lambda(t_1+\kappa^z_{\lambda,\pi})}(i_1) = 0\} \\ \cap \{\exists i_2 \in [\![-\lfloor \lambda^{-z} \rfloor\,, 0]\!], N^S_{\mathbf{a}_\lambda(t_1+\kappa^z_{\lambda,\pi})}(i_2) = 0\}.$$

First, Lemma II.4.4 together with Lemma II.7.1-1 show that $\mathbb{P}\left[\tilde{\Omega}^{P,z}_{\lambda,\pi}(0,t_1)\right]$ tends to 1 when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(\infty, z_0)$ (because $t_1 + \kappa^z_{\lambda,\pi} < (z+t_1)/2 < z$ for (λ, π) sufficiently close to the regime $\mathcal{R}(\infty, z_0)$). Next, on $\tilde{\Omega}^{P,z}_{\lambda,\pi}(0,t_1)$, there holds that

$$C(\vartheta^\lambda_{t_1+\kappa^z_{\lambda,\pi}}, 0) := [\![C^-\,, C^+]\!] \subset [\![-\lfloor \lambda^{-z} \rfloor\,, \lfloor \lambda^{-z} \rfloor]\!].$$

Since C^+ and C^- are vacant during $[\mathbf{a}_\lambda t_1\,, \mathbf{a}_\lambda(t_1+\kappa^z_{\lambda,\pi})] \subset [0\,, \mathbf{a}_\lambda(t_1+\kappa^z_{\lambda,\pi})]$, there holds that, as seen in **Micro**(∞, z_0) in Subsection II.4.4,

$$C^P((\zeta^{\lambda,\pi}_t(i))_{t\geq 0, i\in\mathbb{Z}}, (0,t_1)) \subset C(\vartheta^\lambda_{t_1+\kappa^z_{\lambda,\pi}}, 0) \subset [\![-\lfloor \lambda^{-z} \rfloor\,, \lfloor \lambda^{-z} \rfloor]\!]$$

and $\zeta^{\lambda,\pi}_{\mathbf{a}_\lambda(t_1+\kappa^z_{\lambda,\pi})}(i) \leq 1$ for all $i \in \mathbb{Z}$. Besides, $C^P((\zeta^{\lambda,\pi}_t(i))_{t\geq 0, i\in\mathbb{Z}}, (0,t_1))$ clearly contains $C(\vartheta^\lambda_{t_1}, 0)$, see Figure II.5.

We trivially deduce that, conditionaly on $\tilde{\Omega}^{P,z}_{\lambda,\pi}(0,t_1)$,

$$t_1 + \Xi^\lambda_{t_1} \leq t_1 + \Theta^{\lambda,\pi}_{t_1} \leq t_1 + \kappa^z_{\lambda,\pi} + \Xi^\lambda_{t_1+\kappa^z_{\lambda,\pi}}.$$

Remark now that the function $: t \mapsto t + \Xi^\lambda_t$ is a.s. non decreasing and right-continuous. We thus deduce from Step 1 that

$$t_1 + \Theta^{\lambda,\pi}_{t_1} \xrightarrow[\lambda\to 0]{} 2t_1$$

in probability, whence for all $\delta > 0$ and all $\varepsilon > 0$, there holds that $\mathbb{P}\left[|\Theta^{\lambda,\pi}_{t_1} - t_1| \geq \delta\right] < \varepsilon$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(\infty, z_0)$. □

II.7.3. Proof of Theorem II.6.2

Let us fix $z_0 \in [0\,, 1]$, $x_0 \in (-A\,, A)$, $t_0 > 0$ and $\varepsilon > 0$. The aim of this Section is to prove the

Lemma II.7.3. *For all $\delta > 0$, there holds that*

$$\mathbb{P}\left[\boldsymbol{\delta}(D^{\lambda,\pi,A}_{t_0}(x_0), D^A_{t_0}(x_0)) > \varepsilon\right] < \delta, \tag{II.7.2}$$

$$\mathbb{P}\left[\boldsymbol{\delta}_T(D^{\lambda,\pi,A}(x_0), D^A(x_0)) > \varepsilon\right] < \delta, \tag{II.7.3}$$

for all (λ, π) sufficiently close to the regime $\mathcal{R}(\infty, z_0)$.

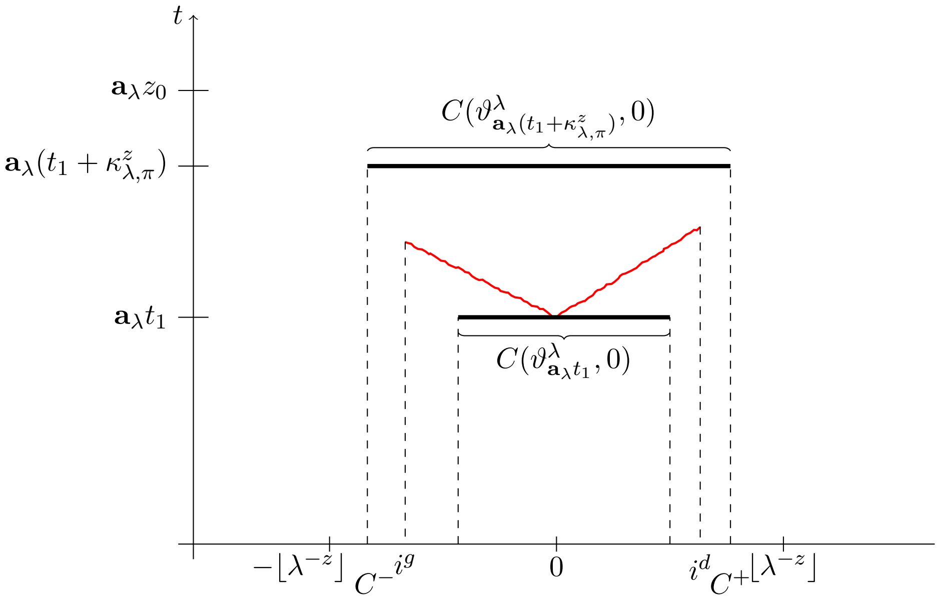

Figure II.5.: Height of a barrier: the true killed cluster.

A match falls in 0 at time $\mathbf{a}_\lambda t_1$. The dashed verticals lines represent vacant sites. The zones $C(\vartheta^\lambda_{\mathbf{a}_\lambda t_1},0)$ and $C(\vartheta^\lambda_{\mathbf{a}_\lambda(t_1+\varkappa^z_{\lambda,\pi})},0)$ are delimited by vacant sites. The site i^g is the first non-positive site where $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_1+T^1_i}(i)=0$ and i^d is the first non-negative site where $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_1+T^1_i}(i)=0$. On $\tilde{\Omega}^{P,z}_{\lambda,\pi}(0,t_1)$, there holds that $-\lfloor\lambda^{-z}\rfloor<i^g<0<i^d<\lfloor\lambda^{-z}\rfloor$ and there is no spark in $[\![i^g\,,i^d]\!]$. The slope lines represent the burning sites.
Finally, the true destroyed component is included in $C(\vartheta^\lambda_{\mathbf{a}_\lambda(t_1+\varkappa^z_{\lambda,\pi})},0)$ but contains $C(\vartheta^\lambda_{\mathbf{a}_\lambda t_1},0)$.

Clearly, (II.7.2) and (II.7.3) will imply the result. Let us first show that (II.7.2) (which holds for an arbitrary value of $t_0\in(0\,,T)$) implies (II.7.3). Indeed, we have by construction for any $t\in[0\,,T]$, $\boldsymbol{\delta}(D^{\lambda,\pi,A}_t(x_0),D^A_t(x_0))<4A$. Hence, by dominated convergence, (II.7.2) implies that $\mathbb{E}\left[\boldsymbol{\delta}(D^{\lambda,\pi,A}_t(x_0),D^A_t(x_0)\right]<\delta$ for all (λ,π) sufficiently close to the regime $\mathcal{R}(\infty,z_0)$, whence again by dominated convergence, $\mathbb{E}\left[\boldsymbol{\delta}_T(D^{\lambda,\pi,A}(x_0),D^A(x_0))\right]<\delta$.

II.7.3.1. The coupling

We are going to construct a coupling between the $(\lambda,\pi,A)-$FFP (on the time interval $[0\,,\mathbf{a}_\lambda T]$) and the LFFP$(\infty,z_0)$ (on $[0\,,T]$): we build the LFFP(∞,z_0) $(Y_t(x))_{t\in[0,T],x\in[-A,A]}$ from a Poisson measure π_M and we take for the matches for the discrete process the Poisson process

$$N^M_t(i)=\pi_M([i/\mathbf{n}_\lambda\,,(i+1)/\mathbf{n}_\lambda)\times[0\,,t/\mathbf{a}_\lambda])$$

for all $i \in I_A^\lambda$ and $t \in [0\,,\mathbf{a}_\lambda T]$.

We next introduce a family of i.i.d. Poisson processes $(N_t^S(i))_{t\geq 0, i\in\mathbb{Z}}$ and $(N_t^P(i))_{t\geq 0, i\in\mathbb{Z}}$ with respective parameter 1 and π, independent of π_M.

The $(\lambda, \pi, A)-$FFP $(\eta_t^{\lambda,\pi}(i))_{t\geq 0, i\in I_A^\lambda}$ is built from the seed processes $(N_t^S(i))_{t\geq 0, i\in\mathbb{Z}}$, from the match processes $(N_t^M(i))_{t\geq 0, i\in\mathbb{Z}}$ and from the propagation processes $(N_t^P(i))_{t\geq 0, i\in\mathbb{Z}}$.

Observe that $(Y_t(x))_{t\in[0,T], x\in[-A,A]}$ is independent of $(N_t^S(i))_{t\in[0,\mathbf{a}_\lambda T], i\in I_A^\lambda}$ and $(N_t^P(i))_{t\in[0,\mathbf{a}_\lambda T], i\in I_A^\lambda}$.

When a match falls at some $x \in [-A\,,A]$ at some time $t \in [0\,,T]$ for the LFFP(∞, z_0), it will fall at $\lfloor \mathbf{n}_\lambda x \rfloor$ at time $\mathbf{a}_\lambda t$ in the discrete process.

II.7.3.2. A sweet event

We call

$$n := \pi_M([0,T] \times [-A\,,A])$$

and we consider the marks $(T_q, X_q)_{q=1,\dots,n}$ of π_M ordered in such a way that $0 < T_1 < \dots < T_n < T$. We introduce

$$\mathcal{T}_M = \{T_1, \dots, T_n\} \text{ and } \mathcal{B}_M = \{X_1, \dots, X_n\}.$$

We also introduce

$$\mathcal{S}_M = \{2t : t \in \mathcal{T}_M, t < z_0\}\,,$$

which has to be seen as the possible limit values of $t + \Theta_t^{\lambda,\pi} \simeq t + t$, recall Lemma II.7.2.

For $\alpha > 0$, we consider the event

$$\Omega_M^0(\alpha) = \Big\{ \min_{\substack{s\in\mathcal{T}_M\cup\mathcal{S}_M,\\ t\in\{0,z_0,t_0\}}} |t-s| > 2\alpha, \min_{\substack{x,y\in\mathcal{B}_M\cup\{x_0,-A,A\},\\ x\neq y}} |x-y| > 2\alpha \Big\},$$

which clearly satisfies $\lim_{\alpha\to 0} \mathbb{P}\left[\Omega_M^0(\alpha)\right] = 1$. For any given $\alpha \in (0\,,1)$, on $\Omega_M^0(\alpha)$, there holds that for all $x, y \in \mathcal{B}_M \cup \{x_0\}$ with $x \neq y$, $(x)_\lambda^\alpha \cap (y)_\lambda^\alpha = \emptyset = (x)_\lambda \cap (y)_\lambda$.

We set

$$z_\alpha = (z_0 - \alpha) \vee (z_0/2).$$

For $q \in \{1, \dots, n\}$, using the seed processes family $(N_t^S(i))_{t\geq 0, i\in\mathbb{Z}}$ and the propagation processes family $(N_t^P(i))_{t\geq 0, i\in\mathbb{Z}}$, we build, recall Definition II.4.6, $(\check{\zeta}_t^{\lambda,\pi,q}(i))_{t\geq 0, i\in\mathbb{Z}}$ the propagation process ignited at (X_q, T_q), $(i_t^{q,+})_{t\geq 0}$ and $(i_t^{q,-})_{t\geq 0}$ the corresponding right and left fronts, and $(T_i^q)_{i\in\mathbb{Z}}$ the associated burning times. We also define $\Omega_{\lambda,\pi}^{P,T,\alpha}(X_q, T_q)$ and $\Omega_{\lambda,\pi}^{P,z_\alpha}(X_q, T_q)$, recall Definition II.4.7. If $z_0 \in (0\,,1]$, we set

$$\Omega^{P,T}(\alpha, \lambda, \pi) = \bigcap_{q=1,\dots,n} (\Omega_{\lambda,\pi}^{P,T,\alpha}(X_q, T_q) \cap \Omega_{\lambda,\pi}^{P,z_\alpha}(X_q, T_q)).$$

If $z_0 = 0$, we simply set

$$\Omega^{P,T}(\alpha, \lambda, \pi) = \bigcap_{q=1,\dots,n} \Omega_{\lambda,\pi}^{P,T,\alpha}(X_q, T_q).$$

By Lemma II.4.4 and since π_M is independent of $(N_t^S(i))_{t\geq 0, i\in\mathbb{Z}}$ and $(N_t^P(i))_{t\geq 0, i\in\mathbb{Z}}$, we deduce that $\mathbb{P}\left[\Omega^{P,T}(\alpha,\lambda,\pi)\right]$ tends to 1 when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(\infty, z_0)$.

Next we introduce the event $\Omega_1^S(\lambda,\pi)$ on which the following conditions hold: for all $q \in \{1,\dots,n\}$,

- if $T_q < z_\alpha$, there are $-\lfloor \lambda^{-z_\alpha} \rfloor < i_1^q < 0 < i_2^q < \lfloor \lambda^{-z_\alpha} \rfloor$ with $N^S_{\mathbf{a}_\lambda(T_q+\kappa^{z_\alpha}_{\lambda,\pi})}(\lfloor \mathbf{n}_\lambda X_q \rfloor + i_1^q) = N^S_{\mathbf{a}_\lambda(T_q+\kappa^{z_\alpha}_{\lambda,\pi})}(\lfloor \mathbf{n}_\lambda X_q \rfloor + i_2^q) = 0$;
- if $T_q > z_0 + \alpha$, for all $i \in (X_q)^\alpha_\lambda$, $N^S_{\mathbf{a}_\lambda T_q}(i) > 0$.

Since $\kappa^{z_\alpha}_{\lambda,\pi}$ can be made arbitrarily small in the regime $\mathcal{R}(\infty, z_0)$, Lemma II.7.1 then show that $\mathbb{P}\left[\Omega_1^S(\lambda,\pi)\right]$ tends to 1 when $\lambda \to 0$ and $\to \infty$ in the regime $\mathcal{R}(\infty, z_0)$.

We also consider the event $\Omega_2^S(\lambda)$ on which the following conditions holds

- if $t_0 < 1$, there are $\lfloor \mathbf{n}_\lambda x_0 \rfloor - \mathbf{m}_\lambda < i_1^0 < \lfloor \mathbf{n}_\lambda x_0 \rfloor < i_2^0 < \lfloor \mathbf{n}_\lambda x_0 \rfloor + \mathbf{m}_\lambda$ with $N^S_{\mathbf{a}_\lambda t_0}(i_1) = N^S_{\mathbf{a}_\lambda t_0}(i_2) = 0$;
- for all $i \in [\![-A_\lambda\,, A_\lambda]\!]$, $N^S_{\mathbf{a}_\lambda}(i) > 0$.

Lemma II.7.1 together with space/time stationarity implies that $\lim_{\lambda\to 0} \mathbb{P}\left[\Omega_2^S(\lambda)\right] = 1$.

We also need $\Omega_3^{S,P}(\gamma,\lambda,\pi)$, defined for $\gamma > 0$ as follows: for all $q = 1,\dots,n$ with $T_q < z_0$, there holds that $|\Theta^{\lambda,\pi,q}_{T_q} - T_q| < \gamma$. Here $\Theta^{\lambda,\pi,q}_{T_q}$ is defined as in Lemma II.7.2 with the seed processes family $(N_t^{S,q}(i))_{t\geq 0, i\in\mathbb{Z}} = (N_t^S(i+\lfloor \mathbf{n}_\lambda X_q \rfloor))_{t\geq 0, i\in\mathbb{Z}}$ and the propagation processes family $(N_t^{P,q}(i))_{t\geq 0, i\in\mathbb{Z}} = (N_t^P(i+\lfloor \mathbf{n}_\lambda X_q \rfloor))_{t\geq 0, i\in\mathbb{Z}}$. Lemma II.7.2 directly implies that for any $\gamma > 0$, $\mathbb{P}\left[\Omega_3^{S,P}(\gamma,\lambda,\pi)\right]$ tends to 1 when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(\infty, z_0)$.

We finally introduce the event

$$\Omega(\alpha,\gamma,\lambda,\pi) = \Omega_M^0(\alpha) \cap \Omega^{P,T}(\alpha,\lambda,\pi) \cap \Omega_1^S(\lambda,\pi) \cap \Omega_2^S(\lambda) \cap \Omega_3^{S,P}(\gamma,\lambda,\pi).$$

We have shown that for any $\delta > 0$, there exists $\alpha \in (0\,,1)$ such that for any $\gamma > 0$, there holds $\mathbb{P}\left[\Omega(\alpha,\gamma,\lambda,\pi)\right] > 1 - \delta$ for all (λ,π) sufficiently close to the regime $\mathcal{R}(\infty, z_0)$.

II.7.3.3. Heart of the proof

The next Lemma is the key of the proof: it guarantees that each fire have a local effect. It will be repeteadly used in Lemmas II.7.5 and II.7.6.

Lemma II.7.4. *On $\Omega(\alpha,\gamma,\lambda,\pi)$, the match falling on $\lfloor \mathbf{n}_\lambda X_q \rfloor$ at time $\mathbf{a}_\lambda T_q$, for some $q \in \{1,\dots,n\}$, does not affect the zone outside $(X_q)^\alpha_\lambda$ during $[\mathbf{a}_\lambda T_q\,, \mathbf{a}_\lambda T]$.*

Consequently, on $\Omega(\alpha,\gamma,\lambda,\pi)$, for all $i \in I_A^\lambda \setminus \cup_{q=1,\dots,n}(X_q)^\alpha_\lambda$ and all $t \in [0\,,T]$, there holds that

$$\eta^{\lambda,\pi,A}_{\mathbf{a}_\lambda t}(i) = \min(N^S_{\mathbf{a}_\lambda t}(i), 1).$$

Proof. As be seen in $\mathbf{Macro}(\infty, z_0)$ in Subsection II.4.4, on $\Omega^{P,T,\alpha}_{\lambda,\pi}(X_q, T_q) \subset \Omega(\alpha,\gamma,\lambda,\pi)$, there holds that

$$X_q - \frac{\mathbf{m}^\alpha_\lambda}{\mathbf{n}_\lambda} \le \frac{\lfloor \mathbf{n}_\lambda X_q \rfloor + i^{q,-}_{\mathbf{a}_\lambda T}}{\mathbf{n}_\lambda} \le X_q \le \frac{\lfloor \mathbf{n}_\lambda X_q \rfloor + 1 + i^{q,+}_{\mathbf{a}_\lambda T}}{\mathbf{n}_\lambda} \le X_q + \frac{\mathbf{m}^\alpha_\lambda}{\mathbf{n}_\lambda}$$

with $\mathbf{m}^\alpha_\lambda / \mathbf{n}_\lambda \le \alpha$. Hence, each fire has only a local effect and does not affect the zone outside $(X_q)^\alpha_\lambda$. □

We now turn to fires of the second kind.

Lemma II.7.5. *Let $q \in \{1, \dots, n\}$ such that $T_q > z_0 + \alpha$. On $\Omega(\alpha,\gamma,\lambda,\pi)$, for all $t \in [\mathbf{a}_\lambda T_q \,, \mathbf{a}_\lambda T]$, there holds that*

$$\eta^{\lambda,\pi,A}_{\mathbf{a}_\lambda t}(\lfloor \mathbf{n}_\lambda X_q \rfloor + i^{q,+}_{\mathbf{a}_\lambda (t-T_q)}) = 2 = \eta^{\lambda,\pi,A}_{\mathbf{a}_\lambda t}(\lfloor \mathbf{n}_\lambda X_q \rfloor + i^{q,-}_{\mathbf{a}_\lambda (t-T_q)}).$$

Proof. At time $\mathbf{a}_\lambda T_q -$, at least one seed has fallen on each site of $(X_q)^\alpha_\lambda$, thanks to $\Omega^S_1(\lambda,\pi)$. Thus, the zone $(X_q)^\alpha_\lambda$ is completely filled at time $\mathbf{a}_\lambda T_q -$, thanks to Lemma II.7.4 (no fire can affect this zone during $[0 \,, \mathbf{a}_\lambda T_q)$). The conclusion is then straightforward, since on $\Omega^{P,T}_{\lambda,\pi}(X_q, T_q)$ there holds that $i^{q,+} \le \mathbf{m}^\alpha_\lambda / \mathbf{n}_\lambda$ and $i^{q,-} \le \mathbf{m}^\alpha_\lambda / \mathbf{n}_\lambda$ (as seen in $\mathbf{Macro}(\infty, z_0)$ in Subsection II.4.4) and since no match falling outside $(X_q)^\alpha_\lambda$ can affect this zone. □

Finally, we treat the case of the fires of the first kind.

Lemma II.7.6. *Let $q \in \{1, \dots, n\}$ such that $T_q < z_0 - \alpha$. On $\Omega(\alpha,\gamma,\lambda,\pi)$, there holds that*

$$(\eta^{\lambda,\pi,A}_{\mathbf{a}_\lambda t}(i))_{t \in [0,T], i \in (X_q)^\alpha_\lambda} = (\zeta^{\lambda,\pi,q}_{T_q,t}(i - \lfloor \mathbf{n}_\lambda X_q \rfloor))_{t \in [0,T], i \in (X_q)^\alpha_\lambda},$$

where the last process is defined as in Lemma II.7.2, using the seed processes family $(N^{S,q}_t(i))_{t \ge 0, i \in \mathbb{Z}} = (N^S_t(i + \lfloor \mathbf{n}_\lambda X_q \rfloor))_{t \ge 0, i \in \mathbb{Z}}$ and the propagation processes family $(N^{P,q}_t(i))_{t \ge 0, i \in \mathbb{Z}} = (N^P_t(i + \lfloor \mathbf{n}_\lambda X_q \rfloor))_{t \ge 0, i \in \mathbb{Z}}$.

Consequently, on $\Omega(\alpha,\gamma,\lambda,\pi)$, for some $\gamma \in (0 \,, \alpha)$,

(a) if $t \in [T_q + \alpha \,, 2T_q - \alpha]$, then there exists $i \in (X_q)^\alpha_\lambda$ such that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}(i) = 0$,

(b) if $t \ge (2T_q + \alpha) \vee 1$, then $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}(i) = 1$ for all $i \in (X_q)^\alpha_\lambda$.

Proof. First observe that the process $(\eta^{\lambda,\pi,A}_{\mathbf{a}_\lambda t}(\lfloor \mathbf{n}_\lambda X_q \rfloor + i))_{t \in [0,T], i \in [\![-\mathbf{m}^\alpha_\lambda, \mathbf{m}^\alpha_\lambda]\!]}$ and the process $(\zeta^{\lambda,\pi,q}_{T_q,t}(i))_{t \in [0,T], i \in [\![-\mathbf{m}^\alpha_\lambda, \mathbf{m}^\alpha_\lambda]\!]}$ evolve according to the same seed processes family and to the same propagation processes family.

Lemma II.7.4 implies that, for all $i \in (X_q)^\alpha_\lambda$ and all $t \in [0 \,, T_q)$,

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}(i) = \min(N^S_{\mathbf{a}_\lambda t}(i), 1),$$

because no match falls in $(X_q)^\alpha_\lambda$ during $[0\,,\mathbf{a}_\lambda T_q)$. This in particular implies that, for all $i \in (X_q)^\alpha_\lambda$ and all $t \in [0\,,T_q)$,

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}(i) = \zeta^{\lambda,\pi,q}_{T_q,t}(i - \lfloor \mathbf{n}_\lambda X_q \rfloor).$$

On $\Omega^{P,z_\alpha}_{\lambda,\pi}(X_q,T_q) \cap \Omega^S_1(\lambda,\pi)$, as seen in $\mathbf{Micro}(\infty, z_0)$ in Subsection II.4.4, since the two processes are building using the same seed processes family and the same propagation processes family, there also holds true that for all $i \in (X_q)^\alpha_\lambda$ and all $t \in [T_q\,, T_q + \kappa^{z_\alpha}_{\lambda,\pi}]$,

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}(i) = \zeta^{\lambda,\pi,q}_{T_q,t}(i - \lfloor \mathbf{n}_\lambda X_q \rfloor).$$

Finally, since there is no more burning tree in $(X_q)^\alpha_\lambda$ at time $\mathbf{a}_\lambda(T_q + \kappa^{z_\alpha}_{\lambda,\pi})$ and since seeds fall according to the same processes, we deduce that, thanks again to Lemma II.7.4, the two processes remain equal during $(T_q + \kappa^{z_\alpha}_{\lambda,\pi}\,, T]$.

All this implies that

$$(\eta^{\lambda,\pi,A}_{\mathbf{a}_\lambda t}(i))_{t\in[0,T], i \in (X_q)^\alpha_\lambda} = (\zeta^{\lambda,\pi,q}_{T_q,t}(i - \lfloor \mathbf{n}_\lambda X_q \rfloor))_{t\in[0,T], i \in (X_q)^\alpha_\lambda}. \tag{II.7.4}$$

Consider now the zone destroyed by the match falling on $\lfloor \mathbf{n}_\lambda X_q \rfloor$ at time $\mathbf{a}_\lambda T_q$

$$C^P \coloneqq C^P((\eta^{\lambda,\pi}_t(i))_{t\geq 0, i\in\mathbb{Z}}, (X_q,T_q)).$$

As seen in $\mathbf{Micro}(\infty, z_0)$ in Subsection II.4.4, $C^P \subset [\![-\lfloor \lambda^{-z_\alpha} \rfloor\,, \lfloor \lambda^{-z_\alpha} \rfloor]\!]$ because there are $i_1 \in [\![-\lfloor \lambda^{-z_\alpha} \rfloor\,, 0]\!]$ and $i_2 \in [\![0\,, \lfloor \lambda^{-z_\alpha} \rfloor]\!]$ which are vacant until $\mathbf{a}_\lambda(T_q + \kappa^{z_\alpha}_{\lambda,\pi})$, thanks to $\Omega^S_1(\lambda,\pi)$.

From (II.7.4) and since no match falling outside $(X_q)^\alpha_\lambda$ can affect this zone, it follows that

$$\Theta^{\lambda,\pi,q}_{T_q} = \inf \left\{ t > T_q : \forall i \in C^P((\eta^{\lambda,\pi}_t(i))_{t\geq 0, i\in\mathbb{Z}}, (X_q,T_q)), \eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}(i) = 1 \right\}.$$

Hence, the zone C^P is not completely occupied during $(\mathbf{a}_\lambda(T_q + \kappa^z_{\lambda,\pi})\,, \mathbf{a}_\lambda(T_q + \Theta^{\lambda,\pi,q}_{T_q}))$ but is completely filled at time $\mathbf{a}_\lambda(T_k + \Theta^{\lambda,\pi,q}_{T_q})$.

Using $\Omega^{S,P}_3(\gamma,\lambda,\pi) \cap \Omega^0_M(\alpha)$ and since $\gamma \in (0\,,\alpha)$, we deduce that,

$$T_q + \alpha < 2T_q - \alpha \leq 2T_q - \gamma \leq T_q + \Theta^{\lambda,\pi,q}_{T_q} \leq 2T_q + \gamma \leq 2T_q + \alpha.$$

We now conclude.

(a) If $t \in [T_q + \alpha\,, 2T_q - \alpha]$, then the zone C^P is not completely occupied at time t. Hence, there exists $i \in C^P \subset (X_q)^\alpha_\lambda$ such that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}(i) = 0$.

(b) If $t \geq (2T_q + \alpha) \vee 1$, then C^P is completely filled at time t because $t \geq T_q + \alpha$.

Consider now $i \in (X_q)^\alpha_\lambda \setminus C^P$. Then i has not been killed by the fire starting at $\lfloor \mathbf{n}_\lambda X_q \rfloor$. Thus i cannot have been killed during $[0\,, \mathbf{a}_\lambda t] \supset [0\,, \mathbf{a}_\lambda]$, thanks to Lemma II.7.4. We conclude using that $t \geq 1$, so that on $\Omega^S_1(\lambda)$, i is occupied at time $\mathbf{a}_\lambda t$. □

II.7.3.4. Conclusion

First, the case $t_0 < 1$ is simple.

Lemma II.7.7. *For $t_0 < 1$, on $\Omega(\alpha, \gamma, \lambda, \pi)$, there holds that*

$$\boldsymbol{\delta}(D_{t_0}^{\lambda,\pi,A}(x_0), D_{t_0}^{A}(x_0)) < \frac{2\mathbf{m}_\lambda}{\mathbf{n}_\lambda}.$$

Proof. Thanks to $\Omega_2^S(\lambda)$, there are $i_1^0 \in [\![\lfloor \mathbf{n}_\lambda x_0 \rfloor - \mathbf{m}_\lambda\, ,\, \lfloor \mathbf{n}_\lambda x_0 \rfloor]\!]$ and $i_2^0 \in [\![\lfloor \mathbf{n}_\lambda x_0 \rfloor\, ,\, \lfloor \mathbf{n}_\lambda x_0 \rfloor + \mathbf{m}_\lambda]\!]$ such that $\eta_{\mathbf{a}_\lambda t_0}^{\lambda,\pi,A}(i_1) = \eta_{\mathbf{a}_\lambda t_0}^{\lambda,\pi,A}(i_2) = 0$. Thus, $C(\eta_{\mathbf{a}_\lambda t_0}^{\lambda,\pi}, \lfloor \mathbf{n}_\lambda x_0 \rfloor) \subset [\![\lfloor \mathbf{n}_\lambda x_0 \rfloor - \mathbf{m}_\lambda\, ,\, \lfloor \mathbf{n}_\lambda x_0 \rfloor + \mathbf{m}_\lambda]\!]$ whence $D_{t_0}^{\lambda,\pi,A}(x_0) \subset [x_0 - \mathbf{m}_\lambda/\mathbf{n}_\lambda\, , x_0 + \mathbf{m}_\lambda/\mathbf{n}_\lambda]$. Since $D_{t_0}^{A}(x_0) = \{x_0\}$, we deduce that

$$\boldsymbol{\delta}(D_{t_0}^{\lambda,\pi,A}(x_0), D_{t_0}^{A}(x_0)) \le \frac{2\mathbf{m}_\lambda}{\mathbf{n}_\lambda}. \qquad \square$$

We now turn to the case $t_0 \ge 1$.

Lemma II.7.8. *For $t_0 \ge 1$, on $\Omega(\alpha, \gamma, \lambda, \pi)$ for some $0 < \gamma < \alpha$ and for all (λ, π) sufficiently close to the regime $\mathcal{R}(\infty, z_0)$ in such a way that $\kappa_{\lambda,\pi}^{z\alpha} \le \alpha$ and $\lfloor z^{-\alpha} \rfloor \le \mathbf{m}_\lambda^\alpha$, there holds that*

$$\boldsymbol{\delta}(D_{t_0}^{\lambda,\pi,A}(x_0), D_{t_0}^{A}(x_0)) < \frac{2\mathbf{m}_\lambda^\alpha}{\mathbf{n}_\lambda}.$$

Proof. Clearly, since $t_0 \ge 1$, $D_{t_0}^{A}(x_0) = [a\, , b]$ for some $a, b \in \mathcal{B}_M \cup \{-A, A\}$. Assume $-A < a < b < A$, the other cases being treated similarly. In the limit process, we then have $Y_{t_0}(a) > 0$, $Y_{t_0}(b) > 0$ and $Y_{t_0}(x) = 0$ for all $x \in (a\, , b)$. We will prove separately that

1. there are $i \in (a)_\lambda^\alpha$ and $j \in (b)_\lambda^\alpha$ such that $\eta_{\mathbf{a}_\lambda t_0}^{\lambda,\pi,A}(i) = 0$ or 2 and $\eta_{\mathbf{a}_\lambda t_0}^{\lambda,\pi,A}(j) = 0$ or 2;
2. for all $x \in \mathcal{B}_M \cap (a\, , b)$, for all $i \in (x)_\lambda^\alpha$, $\eta_{\mathbf{a}_\lambda t_0}^{\lambda,\pi,A}(i) = 1$;
3. for all $i \in [\![\lfloor \mathbf{n}_\lambda a \rfloor + \mathbf{m}_\lambda^\alpha + 1\, , \lfloor \mathbf{n}_\lambda b \rfloor - \mathbf{m}_\lambda^\alpha - 1]\!] \setminus \cup_{x \in \mathcal{B}_M \cap (a,b)} (x)_\lambda^\alpha$, we have $\eta_{\mathbf{a}_\lambda t_0}^{\lambda,\pi,A}(i) = 1$.

Points 1., 2. and 3. imply that,

$$[\![\lfloor \mathbf{n}_\lambda a \rfloor + \mathbf{m}_\lambda^\alpha + 1\, , \lfloor \mathbf{n}_\lambda b \rfloor - \mathbf{m}_\lambda^\alpha - 1]\!] \subset C(\eta_{\mathbf{a}_\lambda t_0}^{\lambda,\pi}, \lfloor \mathbf{n}_\lambda x_0 \rfloor) \subset [\![\lfloor \mathbf{n}_\lambda a \rfloor - \mathbf{m}_\lambda^\alpha - 1\, , \lfloor \mathbf{n}_\lambda b \rfloor + \mathbf{m}_\lambda^\alpha + 1]\!]$$

and thus $[a + \mathbf{m}_\lambda^\alpha/\mathbf{n}_\lambda\, , b - \mathbf{m}_\lambda^\alpha/\mathbf{n}_\lambda] \subset D_{t_0}^{\lambda,\pi,A}(x_0) \subset [a - \mathbf{m}_\lambda^\alpha/\mathbf{n}_\lambda\, , b + \mathbf{m}_\lambda^\alpha/\mathbf{n}_\lambda]$, whence,

$$\boldsymbol{\delta}(D_{t_0}^{A}(x_0), D_{t_0}^{\lambda,\pi,A}(x_0)) \le 2\mathbf{m}_\lambda^\alpha/\mathbf{n}_\lambda.$$

We prove 1. Let $k \in \{1, \dots, n\}$ such that $a = X_k$. There are two cases.

Case 1. If $Y_{t_0}(X_k) = 1$ in the limit process, then $t_0 \ge T_k \ge z_0$ whence $t_0 \ge T_k \ge z_0 + 2\alpha$ due to $\Omega_M^0(\alpha)$. We then use Lemma II.7.5 and conclude that there is a burning tree in $(a)_\lambda^\alpha$ at time $\mathbf{a}_\lambda t_0$.

Case 2. If $Y_{t_0}(a) \in (0\, , 1)$ in the limit process, then $T_k \le z_0 \le 1 \le t_0 \le 2T_k$ whence $T_k + 4\alpha \le z_0 + 2\alpha \le t_0 + 2\alpha \le 2T_k$, due to $\Omega_M^0(\alpha)$. We conclude using Lemma II.7.6-(a) that there is a vacant site in $(a)_\lambda^\alpha$ at time $\mathbf{a}_\lambda t_0$.

Similar considerations hold for b.

We prove 2. Let $x \in \mathcal{B}_M \cap (a\,,b)$ and let $k \in \{1,\ldots,n\}$ such that $x = X_k$.

Case 1. If $T_k > t_0$, then no fire has fallen in $(X_k)^\alpha_\lambda$ during $[0\,,\mathbf{a}_\lambda t_0]$. Using $\Omega^S_1(\lambda,\pi)$ and Lemma II.7.4, we conclude that $(X_k)^\alpha_\lambda$ is completely occupied at time $\mathbf{a}_\lambda t_0$ (because no fire can affect this zone).

Case 2. If $T_k \le t_0$, since in the limit process $Y_{t_0}(X_k) = 0$, necessarily $T_k \le z_0 \le t_0$ and $2T_k \le t_0$ whence $T_k \le z_0 - 2\alpha$ and $2T_k \le t_0 - 2\alpha$ due to $\Omega_M(\alpha)$. Lemma II.7.6-(b) concludes this case since $t_0 \ge (2T_q + \alpha) \vee 1$.

We prove 3. Let $i \in [\![\lfloor \mathbf{n}_\lambda a \rfloor + \mathbf{m}^\alpha_\lambda + 1\,, \lfloor \mathbf{n}_\lambda b \rfloor - \mathbf{m}^\alpha_\lambda - 1]\!] \setminus \cup_{j=1,\ldots,n}(X_j)^\alpha_\lambda$, using Lemma II.7.4 and $\Omega^S_2(\lambda)$, we immediately conclude that i is occupied at time $\mathbf{a}_\lambda t_0$. $\square$

We now can conclude.

Proof of Lemma II.7.3. Let $\delta > 0$ be fixed. We first consider $\alpha_0 \in (0\,,\varepsilon/2)$, $\gamma_0 \in (0\,,\alpha_0)$, $\lambda_0 \in (0\,,1]$, $\epsilon_0 > 0$ and $K_0 \ge 1$ such that for all $\lambda \in (0\,,\lambda_0)$, all $\pi \ge 1$ in such a way that $\frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} \ge K_0$ and $\left|\frac{\log(\pi)}{\log(1/\lambda)} - z_0\right| < \epsilon_0$, we have

$$\mathbb{P}\left[\Omega(\alpha_0,\gamma_0,\lambda,\pi)\right] > 1-\delta.$$

Then we consider $\lambda_1 \in (0\,,\lambda_0)$, $K_1 > K_0$ and $\epsilon_1 \in (0\,,\epsilon_0)$ such that for all $\lambda \in (0\,,\lambda_1)$ and all $\pi \ge 1$ in such a way that $\frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} \ge K_1$ and $\left|\frac{\log(\pi)}{\log(1/\lambda)} - z_0\right| < \epsilon_1$, we have

- $2\mathbf{m}_\lambda/\mathbf{n}_\lambda < \varepsilon$,
- $\kappa^{z_\alpha}_{\lambda,\pi} < \alpha$,
- $2\lambda^{-z_\alpha}/\mathbf{n}_\lambda < 2\mathbf{m}^\alpha_\lambda/\mathbf{n}_\lambda < \varepsilon$.

For all $\lambda \in (0\,,\lambda_1)$, all $\pi \ge 1$ in such a way that $\frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} > K_1$ and $\left|\frac{\log(\pi)}{\log(1/\lambda)} - z_0\right| < \epsilon_1$, Lemma II.7.7 implies that, if $t_0 < 1$,

$$\begin{aligned}\mathbb{P}\left[\boldsymbol{\delta}(D^A_{t_0}(x_0), D^{\lambda,\pi,A}_{t_0}(x_0)) > \varepsilon\right] &\le \mathbb{P}\left[\boldsymbol{\delta}(D^A_{t_0}(x_0), D^{\lambda,\pi,A}_{t_0}(x_0)) > \frac{2\mathbf{m}_\lambda}{\mathbf{n}_\lambda}\right]\\ &\le \mathbb{P}\left[\Omega(\alpha_0,\gamma_0,\lambda,\pi)^c\right]\\ &< \delta\end{aligned}$$

while, if $t_0 \ge 1$, Lemma II.7.8 implies that, (since $\alpha \ge \gamma$ and $\alpha \ge \kappa^{z_\alpha}_{\lambda,\pi}$)

$$\begin{aligned}\mathbb{P}\left[\boldsymbol{\delta}(D^A_{t_0}(x_0), D^{\lambda,\pi,A}_{t_0}(x_0)) > \varepsilon\right] &\le \mathbb{P}\left[\boldsymbol{\delta}(D^A_{t_0}(x_0), D^{\lambda,\pi,A}_{t_0}(x_0)) > \frac{2\mathbf{m}^{\alpha_0}_\lambda}{\mathbf{n}_\lambda}\right]\\ &\le \mathbb{P}\left[\Omega(\alpha_0,\gamma_0,\lambda,\pi)^c\right]\\ &< \delta.\end{aligned}$$

This concludes the proof. $\square$

II.8. Convergence in the intermediate regime

The aim of this section is to prove Theorem II.6.1 for $p > 0$ and this will conclude the proof of Theorem II.2.4 for $p > 0$.

In the whole section, we fix the parameters $A > 0$, $T > 2$ and $p > 0$. We omit the subscript/superscript A in the whole proof.

We recall that $\mathbf{a}_\lambda = \log(1/\lambda)$, $\mathbf{n}_\lambda = \lfloor 1/(\lambda \mathbf{a}_\lambda) \rfloor$, $\mathbf{m}_\lambda = \lfloor 1/(\lambda \mathbf{a}_\lambda^2) \rfloor$, $\varepsilon_\lambda = 1/\mathbf{a}_\lambda^3$. We set as usual $A_\lambda = \lfloor \mathbf{n}_\lambda A \rfloor$ and $I_A^\lambda = [\![-A_\lambda\,, A_\lambda]\!]$. For $i \in \mathbb{Z}$, we set $i_\lambda = [i/\mathbf{n}_\lambda\,, (i+1)/\mathbf{n}_\lambda)$. For $[a\,, b]$ an interval of $[-A\,, A]$ and $\lambda \in (0\,,1)$, we introduce, assuming that $-A < a < b < A$,

$$\begin{aligned}
[a\,, b]_\lambda &= [\![\lfloor \mathbf{n}_\lambda a + \mathbf{m}_\lambda \rfloor + 1\,, \lfloor \mathbf{n}_\lambda b - \mathbf{m}_\lambda \rfloor - 1]\!] \subset \mathbb{Z}, \\
[-A\,, b]_\lambda &= [\![-A_\lambda\,, \lfloor \mathbf{n}_\lambda b - \mathbf{m}_\lambda \rfloor - 1]\!] \subset \mathbb{Z}, \\
[a\,, A]_\lambda &= [\![\lfloor \mathbf{n}_\lambda a + \mathbf{m}_\lambda \rfloor + 1\,, A_\lambda]\!] \subset \mathbb{Z}.
\end{aligned}$$

For $\lambda \in (0\,,1)$ and $\pi \geq 1$, we recall that

$$\kappa^0_{\lambda,\pi} = \frac{\mathbf{m}_\lambda}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda$$

and introduce

$$\mathbf{k}_{\lambda,\pi} = \lfloor \mathbf{a}_\lambda \pi \left(\varepsilon_\lambda + \mathfrak{v}_{\lambda,\pi}\right) \rfloor\,, \tag{II.8.1}$$

$$\mathbf{v}_{\lambda,\pi} = \kappa^0_{\lambda,\pi} + \mathfrak{v}_{\lambda,\pi}, \tag{II.8.2}$$

$$\mathbf{e}_{\lambda,\pi} = \varepsilon_\lambda + \mathfrak{v}_{\lambda,\pi}, \tag{II.8.3}$$

where $\mathfrak{v}_{\lambda,\pi} = \left(\frac{T}{p} \vee 2A\right) \left|\frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} - p\right|$. Observe that $\mathbf{k}_{\lambda,\pi}/\mathbf{n}_\lambda$, $\mathbf{v}_{\lambda,\pi}$ and $\mathbf{e}_{\lambda,\pi}$ tend to 0 as $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$.

For $x \in (-A\,, A)$, $\lambda \in (0\,,1)$ and $\pi \geq 1$, we introduce

$$(x)_\lambda = [\![\lfloor \mathbf{n}_\lambda x \rfloor - \mathbf{m}_\lambda\,, \lfloor \mathbf{n}_\lambda x \rfloor + \mathbf{m}_\lambda]\!] \subset \mathbb{Z}, \tag{II.8.4}$$

$$\langle x \rangle_{\lambda,\pi} = [\![\lfloor \mathbf{n}_\lambda x \rfloor - \mathbf{k}_{\lambda,\pi}\,, \lfloor \mathbf{n}_\lambda x \rfloor + \mathbf{k}_{\lambda,\pi}]\!] \subset \mathbb{Z}, \tag{II.8.5}$$

$$[x]_{\lambda,\pi} = [\![\lfloor \mathbf{n}_\lambda x \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}\,, \lfloor \mathbf{n}_\lambda x \rfloor + \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}]\!] \subset \mathbb{Z}. \tag{II.8.6}$$

II.8.1. Occupation of vacant zone

We start with some easy estimates.

Lemma II.8.1. *Consider a family of i.i.d. Poisson processes $(N^S_t(i))_{t \geq 0, i \in \mathbb{Z}}$. Let $a < b$.*

1. *For $t < 1$, $\lim_{\lambda \to 0} \mathbb{P}\left[\forall i \in [\![\lfloor a\mathbf{m}_\lambda \rfloor\,, \lfloor b\mathbf{m}_\lambda \rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) > 0\right] = 0$;*
2. *For $t \geq 1$, $\lim_{\lambda \to 0} \mathbb{P}\left[\forall i \in [\![\lfloor a\mathbf{m}_\lambda \rfloor\,, \lfloor b\mathbf{m}_\lambda \rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) > 0\right] = 1$;*
3. *For $t < 1$, $\lim_{\lambda \to 0} \mathbb{P}\left[\forall i \in [\![\lfloor a\mathbf{n}_\lambda \rfloor\,, \lfloor b\mathbf{n}_\lambda \rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) > 0\right] = 0$;*

4. *For* $t \geq 1$, $\lim_{\lambda \to 0} \mathbb{P}\left[\forall i \in [\![\lfloor a\mathbf{n}_\lambda \rfloor, \lfloor b\mathbf{n}_\lambda \rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) > 0\right] = 1$*;*

5. *For* $t > 0$, $\lim_{\lambda \to 0} \mathbb{P}\left[\exists i \in [\![\lfloor a\mathbf{n}_\lambda \rfloor, \lfloor b\mathbf{n}_\lambda \rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) > 0\right] = 1$*;*

6. *For* $t > 0$ *and* $\delta > 0$, $\lim_{\lambda \to 0} \mathbb{P}\left[\forall i \in [\![-\lfloor \lambda^{-(t+\delta)} \rfloor, \lfloor \lambda^{-(t+\delta)} \rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) > 0\right] = 0$*;*

7. *For* $t > 0$ *and* $\delta > 0$, $\lim_{\lambda \to 0} \mathbb{P}\left[\forall i \in [\![-\lfloor \lambda^{-(t-\delta)} \rfloor, \lfloor \lambda^{-(t-\delta)} \rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) > 0\right] = 1$*;*

8. *For* $t < 1$, $\lim_{\substack{\lambda \to 0 \\ \pi \to \infty}} \mathbb{P}\left[\forall i \in [\![\lfloor a\mathbf{k}_{\lambda,\pi} \rfloor, \lfloor b\mathbf{k}_{\lambda,\pi} \rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) > 0\right] = 0$ *(when* $\lambda \to 0$ *and* $\pi \to \infty$ *in the regime* $\mathcal{R}(p)$*);*

9. *For* $t \geq 1$, $\lim_{\substack{\lambda \to 0 \\ \pi \to \infty}} \mathbb{P}\left[\forall i \in [\![\lfloor a\mathbf{k}_{\lambda,\pi} \rfloor, \lfloor b\mathbf{k}_{\lambda,\pi} \rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) > 0\right] = 1$ *(when* $\lambda \to 0$ *and* $\pi \to \infty$ *in the regime* $\mathcal{R}(p)$*).*

Proof. This lemma is closely related to Lemma II.7.1. For $r_\lambda \xrightarrow[\lambda \to 0]{} \infty$, we have

$$\mathbb{P}\left[\forall i \in [\![-\lfloor ar_\lambda \rfloor, \lfloor br_\lambda \rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) > 0\right] \simeq (1 - e^{-\mathbf{a}_\lambda t})^{(b-a)r_\lambda} \simeq e^{-(b-a)r_\lambda \lambda^t}.$$

Observe now that

$$\mathbf{m}_\lambda \lambda^t \simeq \frac{\lambda^{t-1}}{\mathbf{a}_\lambda^2} \xrightarrow[\lambda \to 0]{} \begin{cases} \infty & \text{if } t < 1, \\ 0 & \text{if } t \geq 1, \end{cases}$$

from which points 1 and 2 follow, that

$$\mathbf{n}_\lambda \lambda^t \simeq \frac{\lambda^{t-1}}{\mathbf{a}_\lambda} \xrightarrow[\lambda \to 0]{} \begin{cases} \infty & \text{if } t < 1, \\ 0 & \text{if } t \geq 1, \end{cases}$$

which implies points 3 and 4. For the point 5, it suffices to note that, for any $i \in \mathbb{Z}$,

$$\mathbb{P}\left[N^S_{\mathbf{a}_\lambda t}(i) = 0\right] = e^{-\mathbf{a}_\lambda t}.$$

Hence

$$\mathbb{P}\left[\exists i \in [\![\lfloor a\mathbf{n}_\lambda \rfloor, \lfloor b\mathbf{n}_\lambda \rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) > 0\right] \simeq 1 - e^{-\mathbf{a}_\lambda \mathbf{n}_\lambda t(b-a)} \xrightarrow[\lambda \to 0]{} 1.$$

For $t > 0$ and $\delta > 0$, we have

$$\mathbb{P}\left[\forall i \in [\![-\lfloor \lambda^{-(t+\delta)} \rfloor, \lfloor \lambda^{-(t+\delta)} \rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) > 0\right] \simeq e^{-2\lambda^{-\delta}} \xrightarrow[\lambda \to 0]{} 0,$$

which is point 6, while

$$\mathbb{P}\left[\forall i \in [\![-\lfloor \lambda^{-(t-\delta)} \rfloor, \lfloor \lambda^{-(t-\delta)} \rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) > 0\right] \simeq e^{-2\lambda^{\delta}} \xrightarrow[\lambda \to 0]{} 1$$

which is Point 7.

For the two last statement, as $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$, we have, observing that $\mathfrak{v}_{\lambda,\pi} \to 0$,

$$\mathbf{k}_{\lambda,\pi} \lambda^t \simeq \mathbf{a}_\lambda \pi \lambda^t (\varepsilon_\lambda + \mathfrak{v}_{\lambda,\pi}) \simeq \frac{\mathbf{n}_\lambda \lambda^t}{p} (\varepsilon_\lambda + \mathfrak{v}_{\lambda,\pi}) \simeq \frac{\lambda^{t-1}}{\mathbf{a}_\lambda p} \left(1/{\mathbf{a}_\lambda}^3 + \mathfrak{v}_{\lambda,\pi}\right) \xrightarrow[\lambda,\pi]{} \begin{cases} \infty & \text{if } t < 1, \\ 0 & \text{if } t \geq 1. \end{cases} \qquad \square$$

II.8.2. Height of the barrier

We describe here the time needed for a destroyed microscopic cluster to be regenerated. Roughly, we assume that the zone $[\![-\mathbf{m}_\lambda\,,\mathbf{m}_\lambda]\!]$ around 0 has been made vacant at some time $\mathbf{a}_\lambda t_0$. Then we consider the situation where a match falls on 0 at some time $\mathbf{a}_\lambda t_1 \in (\mathbf{a}_\lambda t_0\,, \mathbf{a}_\lambda(t_0+1))$ and we compute the delay needed for the destroyed cluster to be fully regenerated. We have to distinguish two cases.

a) We first consider the case where a match falls on 0 at time $\mathbf{a}_\lambda t_1 \in (0\,, \mathbf{a}_\lambda)$. This case is closely related to Lemma II.7.2.

b) We then consider the case where a fire propagates through $[\![-\mathbf{m}_\lambda\,,\mathbf{m}_\lambda]\!]$ at time $\mathbf{a}_\lambda t_0$ and a match falls on 0 at time $\mathbf{a}_\lambda t_1 \in (\mathbf{a}_\lambda t_0\,, \mathbf{a}_\lambda(t_0+1))$. This case is a little bit different but is proved in the same way as the previous case.

Lemma II.8.2. *Consider two Poisson processes $(N_t^S(i))_{t\geq 0, i\in\mathbb{Z}}$ and $(N_t^P(i))_{t\geq 0, i\in\mathbb{Z}}$ with respective rates 1 and π, all this processes being independent. Consider also $\mathcal{M} := (i_0; t_0, t_1) \in \mathbb{Z}\times(\mathbb{R}_+)^2$ with $|i_0| \in [\![\mathbf{m}_\lambda\,, \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}]\!]$, $t_0 \in \{0\}\cup(1\,,\infty)$ and $t_1 \in (t_0\,, t_0+1)$. For $i \in \mathbb{Z}$ and $t \geq 0$, we consider the process*

$$
\begin{aligned}
\zeta_t^{\lambda,\pi,\mathcal{M}}(i) = {} & \left(1 + \mathbf{1}_{\{t\geq \mathbf{a}_\lambda(t_0-\mathbf{v}_{\lambda,\pi}), i=i_0\}}\right) \times \mathbf{1}_{\{t_0>1\}} \\
& + \mathbf{1}_{\{t\geq \mathbf{a}_\lambda t_1, i=0, \zeta^{\lambda,\pi,\mathcal{M}}_{\mathbf{a}_\lambda t_1 -}(0)=1\}} + \int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda,\pi,\mathcal{M}}(i)=0\}}\,\mathrm{d}N_s^S(i) \\
& + \int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda,\pi,\mathcal{M}}(i+1)=2, \zeta_{s-}^{\lambda,\pi,\mathcal{M}}(i)=1\}}\,\mathrm{d}N_s^P(i+1) \\
& + \int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda,\pi,\mathcal{M}}(i-1)=2, \zeta_{s-}^{\lambda,\pi,\mathcal{M}}(i)=1\}}\,\mathrm{d}N_s^P(i-1) \\
& - 2\int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda,\pi,\mathcal{M}}(i)=2\}}\,\mathrm{d}N_s^P(i).
\end{aligned}
$$

Using the propagation processes $(N_t^P(i))_{t\geq 0, i\in\mathbb{Z}}$, consider the burning times $(T_i^1)_{i\in\mathbb{Z}}$ of the propagation process iginited at $(0, t_1)$, recall Definition II.4.6, and define the destroyed cluster due to the match falling in 0 at time $\mathbf{a}_\lambda t_1$, recall (II.4.14),

$$
C^P((\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t\geq 0, i\in\mathbb{Z}}, (0, t_1)) := [\![i^g\,, i^d]\!].
$$

We finally define the time needed for $C^P((\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t\geq 0, i\in\mathbb{Z}}, (0, t_1))$ to become again occupied

$$
\Theta_{\mathcal{M}}^{\lambda,\pi} := \inf\left\{t > t_1 : \forall i \in C^P((\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t\geq 0, i\in\mathbb{Z}}, (0, t_1)), \zeta_{\mathbf{a}_\lambda t}^{\lambda,\pi,\mathcal{M}}(i) = 1\right\}.
$$

For all $\delta > 0$, there holds that,

$$
\lim_{\lambda,\pi} \mathbb{P}\left[\left|\Theta_{\mathcal{M}}^{\lambda,\pi} - (t_1 - t_0)\right| \geq \delta\right] = 0
$$

when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$.

Let us explain the behaviour of the process $(\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t\geq 0,i\in\mathbb{Z}}$. If $t_0=0$, then the process starts from a vacant initial situation and a match falls on 0 at time $\mathbf{a}_\lambda t_1$. It does not depend on i_0 and since $0<t_1<1$, the zone $[\![-\mathbf{m}_\lambda\,,\mathbf{m}_\lambda]\!]$ is not completely filled at time $\mathbf{a}_\lambda(t_1+\kappa^0_{\lambda,\pi})$, see Lemma II.8.1-1 (and because $\kappa^0_{\lambda,\pi}\to 0$). The process is then governed by the propagation processes $(N_t^P(i))_{t\geq 0,i\in\mathbb{Z}}$ and the seed processes $(N_t^S(i))_{t\geq 0,i\in\mathbb{Z}}$ with the same rules as the $(\lambda,\pi)-$FFP. As seen in **Micro**(p) in Subsection II.4.4, the fire is extinguished at time $\mathbf{a}_\lambda(t_1+\kappa^0_{\lambda,\pi})$.

If $t_0>1$, then the process starts at time 0 from an occupied initial situation, nothing happens until a match falls on i_0 at time $\mathbf{a}_\lambda(t_0-\mathbf{v}_{\lambda,\pi})$. Two fires start: one goes to the left and one goes to the right. Thus, on $\Omega^{P,T}_{\lambda,\pi}(i_0/\mathbf{n}_\lambda,t_0-\mathbf{v}_{\lambda,\pi})$, recall Definition II.4.7, and since

$$\lfloor \mathbf{a}_\lambda\pi(3\mathbf{v}_{\lambda,\pi}-\varepsilon_\lambda)\rfloor\geq 2\mathbf{m}_\lambda+2\mathbf{k}_{\lambda,\pi},$$

recall (II.8.1) and (II.8.2), each site of $[\![-\mathbf{m}_\lambda\,,\mathbf{m}_\lambda]\!]$ burns and extinguishes before $\mathbf{a}_\lambda(t_0+2\mathbf{v}_{\lambda,\pi})$, recall Lemma II.4.2. Hence, the zone $[\![-\mathbf{m}_\lambda\,,\mathbf{m}_\lambda]\!]$ is not completely filled when the match falls on 0 at time $\mathbf{a}_\lambda t_1$, see Lemma II.8.1-1 and because $\mathbf{a}_\lambda(t_0+2\mathbf{v}_{\lambda,\pi})<\mathbf{a}_\lambda t_1<\mathbf{a}_\lambda(t_0+1)$ for all (λ,π) sufficiently close to the regime $\mathcal{R}(p)$.

Proof. The proof is in the same spirit as the proof of Lemma II.7.2. We first define the simplest process with an instantaneous propagation: if a match falls in a cluster, it destroys instantaneously the entire connected component. Secondly, we flank the killed cluster $C^P((\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t\geq 0,i\in\mathbb{Z}},(0,t_1))$ to estimate the time needed to become again occupied, see Figure II.6.

Step 1. Let $\tau_0<\tau_1<\tau_0+1$ be fixed. Put $\vartheta^\lambda_{\tau_0,t}(i)=\min(N^S_{\mathbf{a}_\lambda(\tau_0+t)}(i)-N^S_{\mathbf{a}_\lambda\tau_0}(i),1)$ and $\vartheta^\lambda_{\tau_1,t}(i)=\min(N^S_{\mathbf{a}_\lambda(\tau_1+t)}(i)-N^S_{\mathbf{a}_\lambda\tau_1}(i),1)$ for all $t>0$ and all $i\in\mathbb{Z}$. We define the time needed for the destroyed cluster to be fully regenerated

$$\Xi^\lambda_{\tau_0,\tau_1}=\inf\left\{t>0:\forall i\in C(\vartheta^\lambda_{\tau_0,\tau_1-\tau_0},0),\ \vartheta^\lambda_{\tau_1,t}(i)=1\right\}.$$

Then for all $\delta>0$,

$$\lim_{\lambda\to 0}\mathbb{P}\left[|\Xi^\lambda_{\tau_0,\tau_1}-(\tau_1-\tau_0)|\geq\delta\right]=0.$$

This has been checked in Step 1 of the proof of Lemma II.7.2 when $\tau_0=0$. This of course extends without any difficulty, using time stationarity.

Step 2. Assume $t_0=0$. In that case, the process not depends on i_0. Consider the event $\Omega^{P,T}_{\lambda,\pi}(0,t_1)$, recall Definition II.4.7. We define

$$\tilde\Omega^{P,T,\mathcal{M}}_{\lambda,\pi}=\Omega^{P,T}_{\lambda,\pi}(0,t_1)\cap\{\exists i_1\in[\![-\mathbf{m}_\lambda\,,0]\!],N^S_{\mathbf{a}_\lambda(t_1+\kappa^0_{\lambda,\pi})}(i_1)=0\}$$
$$\cap\{\exists i_2\in[\![0\,,\mathbf{m}_\lambda]\!],N^S_{\mathbf{a}_\lambda(t_1+\kappa^0_{\lambda,\pi})}(i_2)=0\}.$$

Lemma II.4.2 together with Lemma II.8.1-1 show that $\mathbb{P}\left[\tilde{\Omega}^{P,T,\mathcal{M}}_{\lambda,\pi}\right]$ tends to 1 when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$ (because $t_1+\kappa^0_{\lambda,\pi} < (t_1+1)/2 < 1$ for all (λ,π) sufficiently close to the regime $\mathcal{R}(p)$).

Next, on $\tilde{\Omega}^{P,T}_{\lambda,\pi}(0,t_1)$, there holds that

$$C(\vartheta^{\lambda}_{0,t_1+\kappa^0_{\lambda,\pi}},0) \coloneqq [\![C^-\,,C^+]\!] \subset [\![i_1\,,i_2]\!] \subset [\![-\mathbf{m}_\lambda\,,\mathbf{m}_\lambda]\!].$$

Since, by definition, no seed falls on C^+ and on C^- until $\mathbf{a}_\lambda(t_1+\kappa^0_{\lambda,\pi})$ and since we start from a vacant initial situation, we deduce that

$$\zeta^{\lambda,\pi,\mathcal{M}}_t(C^-) = \zeta^{\lambda,\pi,\mathcal{M}}_t(C^+) = 0$$

for all $t \in [0\,,\mathbf{a}_\lambda(t_1+\kappa^0_{\lambda,\pi})] \supset [\mathbf{a}_\lambda t_1\,,\mathbf{a}_\lambda(t_1+\kappa^0_{\lambda,\pi})]$. As seen in **Micro**$(p)$ in Subsection II.4.4, the fire destroys exactly the zone $C^P((\zeta^{\lambda,\pi,\mathcal{M}}_t(i))_{t\geq 0,i\in\mathbb{Z}},(0,t_1))$ and

$$C^P((\zeta^{\lambda,\pi,\mathcal{M}}_t(i))_{t\geq 0,i\in\mathbb{Z}},(0,t_1)) \subset [\![C^-\,,C^+]\!] \subset [\![-\mathbf{m}_\lambda\,,\mathbf{m}_\lambda]\!]$$

with $\zeta^{\lambda,\pi,\mathcal{M}}_{\mathbf{a}_\lambda(t_1+\kappa^0_{\lambda,\pi})}(i) \leq 1$ for all $i\in\mathbb{Z}$ (the fire is extinguished at time $\mathbf{a}_\lambda(t_1+\kappa^0_{\lambda,\pi})$).

Since $C^P((\zeta^{\lambda,\pi,\mathcal{M}}_t(i))_{t\geq 0,i\in\mathbb{Z}},(0,t_1))$ clearly contains $C(\vartheta^\lambda_{0,t_1},0)$, we deduce that, on $\tilde{\Omega}^{P,T,\mathcal{M}}_{\lambda,\pi}$,

$$t_1+\Xi^\lambda_{0,t_1} \leq t_1+\Theta^{\lambda,\pi}_{\mathcal{M}} \leq t_1+\kappa^0_{\lambda,\pi}+\Xi^\lambda_{0,t_1+\kappa^0_{\lambda,\pi}}.$$

Remark now that the function : $t\mapsto t+\Xi^\lambda_{0,t}$ is a.s. non decreasing and right-continuous. We thus deduce from Step 1 that

$$t_1+\Theta^{\lambda,\pi}_{\mathcal{M}} \xrightarrow[\lambda,\pi]{\mathbb{P}} 2t_1$$

in probability, whence for all $\delta>0$ and all $\varepsilon>0$, there holds that $\mathbb{P}\left[\left|\Theta^{\lambda,\pi}_{\mathcal{M}}-t_1\right|\geq\delta\right]<\varepsilon$ for all (λ,π) sufficiently close to the regime $\mathcal{R}(p)$.

Step 3. Assume now $t_0>1$. We may and will assume $i_0\in[\![-\mathbf{m}_\lambda-2\mathbf{k}_{\lambda,\pi}\,,-\mathbf{m}_\lambda]\!]$, by symetry.

Consider the events $\Omega^{P,T}_{\lambda,\pi}(i_0/\mathbf{n}_\lambda,t_0-\mathbf{v}_{\lambda,\pi})$ and $\Omega^{P,T}_{\lambda,\pi}(0,t_1)$, recall Definition II.4.7. We define

$$\begin{aligned}\tilde{\Omega}^{P,T,\mathcal{M}}_{\lambda,\pi} \coloneqq\ & \Omega^{P,T}_{\lambda,\pi}(0,t_1)\cap\Omega^{P,T}_{\lambda,\pi}(i_0/\mathbf{n}_\lambda,t_0-\mathbf{v}_{\lambda,\pi})\\ &\cap\{\exists i_1\in[\![-\mathbf{m}_\lambda\,,0]\!], N^S_{\mathbf{a}_\lambda(t_1+\kappa^0_{\lambda,\pi})}(i_1)-N^S_{\mathbf{a}_\lambda(t_0-\mathbf{v}_{\lambda,\pi})}(i_1)=0\}\\ &\cap\{\exists i_2\in[\![0\,,\mathbf{m}_\lambda]\!], N^S_{\mathbf{a}_\lambda(t_1+\kappa^0_{\lambda,\pi})}(i_2)-N^S_{\mathbf{a}_\lambda(t_0-\mathbf{v}_{\lambda,\pi})}(i_2)=0\}.\end{aligned}$$

Lemma II.4.2 together with Lemma II.8.1-1 directly imply that $\mathbb{P}\left[\tilde{\Omega}^{P,T,\mathcal{M}}_{\lambda,\pi}\right]$ tends to 1 when $\lambda\to 0$ and $\pi\to\infty$ in the regime $\mathcal{R}(p)$ (because $t_1+\kappa^0_{\lambda,\pi}-(t_0-\mathbf{v}_{\lambda,\pi})=$

$t_1 - t_0 + \kappa^0_{\lambda,\pi} + \mathbf{v}_{\lambda,\pi} < (t_1 - t_0 + 1)/2 < 1$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(p)$).

Recall Lemma II.4.2. Since all the sites are occupied at time $\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi})$ and since

$$i_0 + \lfloor \mathbf{a}_\lambda \pi (3\mathbf{v}_{\lambda,\pi} - \varepsilon_\lambda) \rfloor \geq \mathbf{m}_\lambda,$$

on $\Omega^{P,T}_{\lambda,\pi}(i_0/\mathbf{n}_\lambda, t_0 - \mathbf{v}_{\lambda,\pi})$, there is no more burning tree in $[\![-\mathbf{m}_\lambda\,, \mathbf{m}_\lambda]\!]$ at time $\mathbf{a}_\lambda(t_0 + 2\mathbf{v}_{\lambda,\pi})$ nor during the time interval $[\mathbf{a}_\lambda(t_0 + 2\mathbf{v}_{\lambda,\pi})\,, \mathbf{a}_\lambda t_1)$. Thus, the match falling in 0 at time $\mathbf{a}_\lambda t_1$ destroys at least the zone $C(\vartheta^\lambda_{t_0+2\mathbf{v}_{\lambda,\pi},t_1}, 0)$.

Next, on $\tilde{\Omega}^{P,T,\mathcal{M}}_{\lambda,\pi}$, we have

$$C(\vartheta^\lambda_{t_0-\mathbf{v}_{\lambda,\pi},t_1+\kappa^0_{\lambda,\pi}}, 0) \coloneqq [\![C^-\,, C^+]\!] \subset [\![i_1\,, i_2]\!] \subset [\![-\mathbf{m}_\lambda\,, \mathbf{m}_\lambda]\!].$$

Since no seed falls on C^- and on C^+ during $[\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi})\,, \mathbf{a}_\lambda(t_1 + \kappa^0_{\lambda,\pi})]$ and since C^- and C^+ are made vacant during the time interval $[\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi})\,, \mathbf{a}_\lambda(t_0 + 2\mathbf{v}_{\lambda,\pi})]$, thanks to $\Omega^{P,T}_{\lambda,\pi}(i_0/\mathbf{n}_\lambda, t_0 - \mathbf{v}_{\lambda,\pi})$, we deduce that there is no burning tree in $[\![C^-\,, C^+]\!]$ at time $\mathbf{a}_\lambda t_1-$ and

$$\zeta^{\lambda,\pi,\mathcal{M}}_{\mathbf{a}_\lambda t}(C^-) = \zeta^{\lambda,\pi,\mathcal{M}}_{\mathbf{a}_\lambda t}(C^+) = 0 \text{ for all } t \in [t_1\,, t_1 + \kappa^0_{\lambda,\pi}].$$

Hence, as seen in **Micro**(p) in Subsection II.4.4, the match falling on 0 at time $\mathbf{a}_\lambda t_1$ destroys at most the zone $[\![C^-\,, C^+]\!] \subset [\![i_1\,, i_2]\!]$ and there is no more burning tree in $[\![C^-\,, C^+]\!]$ at time $\mathbf{a}_\lambda(t_1 + \kappa^0_{\lambda,\pi})$.

To summarize, on $\tilde{\Omega}^{P,T,\mathcal{M}}_{\lambda,\pi}$, see Figure II.6, we have

$$C(\vartheta^\lambda_{t_0+2\mathbf{v}_{\lambda,\pi},t_1}, 0) \subset C^P((\zeta^{\lambda,\pi,\mathcal{M}}_t(i))_{t\geq 0, i\in\mathbb{Z}}, (0, t_1)) \subset C(\vartheta^\lambda_{t_0-\mathbf{v}_{\lambda,\pi},t_1+\kappa^0_{\lambda,\pi}}, 0) \subset [\![i_1\,, i_2]\!]$$

with additionally $\zeta^{\lambda,\pi,\mathcal{M}}_{\mathbf{a}_\lambda(t_1+\kappa^0_{\lambda,\pi})}(i) \leq 1$ for all $i \in [\![-\mathbf{m}_\lambda\,, \mathbf{m}_\lambda]\!]$.

No fire affect the zone $[\![-\mathbf{m}_\lambda\,, \mathbf{m}_\lambda]\!]$ during $[\mathbf{a}_\lambda(t_1+\kappa^0_{\lambda,\pi})\,, \mathbf{a}_\lambda T]$, thanks to $\Omega^{P,T}_{\lambda,\pi}(i_0/\mathbf{n}_\lambda, t_0 - \mathbf{v}_{\lambda,\pi})$. We deduce that, on $\tilde{\Omega}^{P,T,\mathcal{M}}_{\lambda,\pi}$ and for all (λ, π) sufficiently close to the regime $\mathcal{R}(p)$,

$$t_1 + \Xi^\lambda_{t_0+2\mathbf{v}_{\lambda,\pi},t_1} \leq t_1 + \Theta^{\lambda,\pi}_{\mathcal{M}} \leq t_1 + \kappa^0_{\lambda,\pi} + \Xi^\lambda_{t_0-\mathbf{v}_{\lambda,\pi},t_1+\kappa^0_{\lambda,\pi}}.$$

Then, one easily concludes. The function $s \mapsto t_1 + \Xi^\lambda_{t_0+s,t_1}$ is a.s. non increasing and right-continuous while the function $s \mapsto t_1 + s + \Xi^\lambda_{t_0-s,t_1+s}$ is a.s. non decreasing and right-continuous. Since $\kappa^0_{\lambda,\pi} \to 0$, we thus deduce from Step 1 that

$$t_1 + \Theta^{\lambda,\pi}_{\mathcal{M}} \xrightarrow[\lambda,\pi]{\mathbb{P}} 2t_1 - t_0,$$

as desired. $\square$

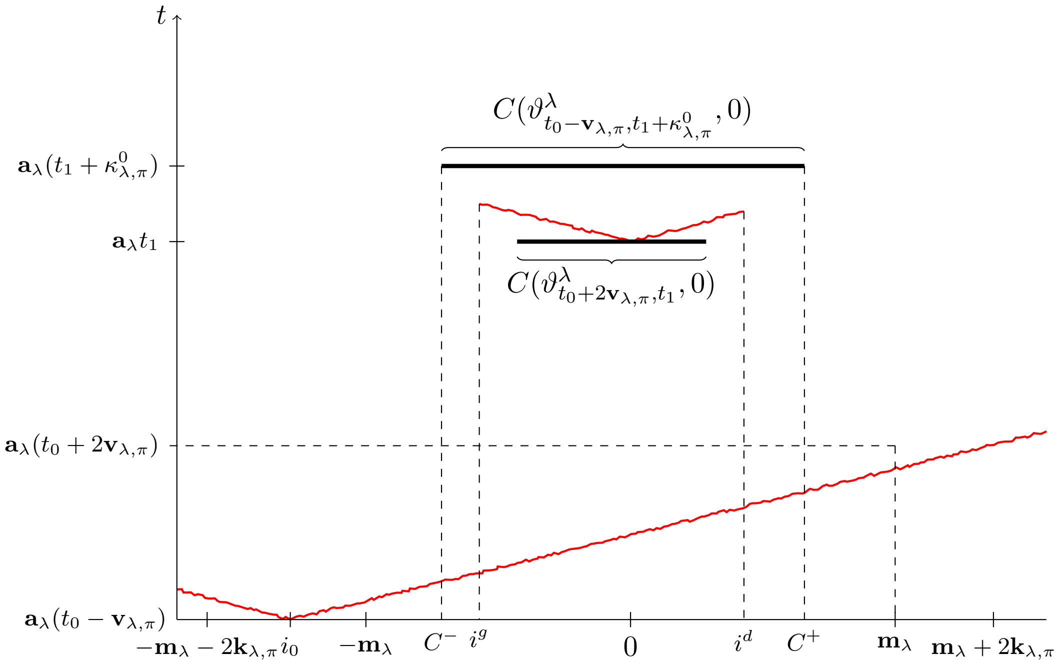

Figure II.6.: Height of a barrier in the regime $\mathcal{R}(p)$, for $p>0$.

At time $\mathbf{a}_\lambda(t_0-\mathbf{v}_{\lambda,\pi})-$, all the sites are occupied. A match falls on i_0 at time $\mathbf{a}_\lambda(t_0-\mathbf{v}_{\lambda,\pi})$. Two fires start: one goes to the left and one goes to the right. Thus, on $\Omega^{P,T}_{\lambda,\pi}(i_0/\mathbf{n}_\lambda,t_0-\mathbf{v}_{\lambda,\pi})$, each site of $[\![-\mathbf{m}_\lambda\,,\mathbf{m}_\lambda]\!]$ burns and extinguishes before $\mathbf{a}_\lambda(t_0+2\mathbf{v}_{\lambda,\pi})$ (because $i_0+\lfloor \mathbf{a}_\lambda\pi(3\mathbf{v}_{\lambda,\pi}-\varepsilon_\lambda)\rfloor\geq\mathbf{m}_\lambda$).
Next, a match falls on 0 at time $\mathbf{a}_\lambda t_1$. Since no seed fall on $C^-\in[\![-\mathbf{m}_\lambda\,,0]\!]$ and $C^+\in[\![0\,,\mathbf{m}_\lambda]\!]$ during $[\mathbf{a}_\lambda(t_0-\mathbf{v}_{\lambda,\pi})\,,\mathbf{a}_\lambda(t_1+\kappa^0_{\lambda,\pi})]$, they remain vacant after burning. Thus, the true killed cluster $[\![i^g\,,i^d]\!]$ contains $C(\vartheta^\lambda_{t_0+2\mathbf{v}_{\lambda,\pi},t_1},0)$ but is included in $[\![C^-\,,C^+]\!]=C^P((\zeta^{\lambda,\pi\mathcal{M}}_t)_{t\geq0,i\in\mathbb{Z}},(0,t_1))$.

II.8.3. Persistent effect of microscopic fires

Here we study the effect of microscopic fires. First, they produce a barrier, and then, if there are alternatively macroscopic fires on the left and right, they still have an effect. This phenomenon is illustrated on Figure II.7 in the case of the limit process.

We say that $\mathcal{P}=(t_0,t_1,\dots,t_K)$ satisfies $(PP1)$ (like ping-pong) if

1. $K\geq2$;
2. $t_0\in\{0\}\cup(1\,,\infty)$ and $t_0<t_1<t_2<\cdots<t_K$;
3. for all $k=0,\dots,K-1$, $t_{k+1}-t_k<1$;
4. $t_2-t_0>1$ and for all $k=2,\dots,K-2, t_{k+2}-t_k>1$.

We say that $\mathcal{I}=(\varepsilon;i_0,i_2,\dots,i_K)$ satisfies $(PP2)$ if

1. $\varepsilon \in \{-1, 1\}$;

2. $|i_0| \in [\![\mathbf{m}_\lambda \,, \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}]\!]$;

3. for all $k = 2, \ldots, K$, $\varepsilon_k i_k \in [\![\mathbf{m}_\lambda \,, \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}]\!]$, where we set $\varepsilon_k = (-1)^k \varepsilon$.

Finally, we say that $\mathfrak{P} = (\mathcal{P}, \mathcal{I})$ satisfies (PP) if $\mathcal{P}$ satisfies $(PP1)$ and $\mathcal{I}$ satisfies $(PP2)$.

Let $\mathfrak{P}$ satisfy (PP). Consider two Poisson processes $(N_t^S(i))_{t\geq 0, i\in\mathbb{Z}}$ and $(N_t^P(i))_{t\geq 0, i\in\mathbb{Z}}$ with respective rates 1 and π, all this processes being independent. We define the process $(\zeta_t^{\lambda,\pi,\mathfrak{P}}(i))_{t\geq 0, i\in\mathbb{Z}}$ as follows

$$
\begin{aligned}
\zeta_t^{\lambda,\pi,\mathfrak{P}}(i) =& (1 + \mathbf{1}_{\{i=i_0, t\geq \mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi})\}})\mathbf{1}_{\{t_0 \geq 1\}} + \mathbf{1}_{\{i=0, t\geq \mathbf{a}_\lambda t_1, \zeta^{\lambda,\pi,\mathfrak{P}}_{\mathbf{a}_\lambda t_1 -}(0)=1\}} \\
&+ \sum_{k=2}^{K} \mathbf{1}_{\{i=i_k, t\geq \mathbf{a}_\lambda(t_k - \mathbf{v}_{\lambda,\pi}), \zeta^{\lambda,\pi,\mathfrak{P}}_{\mathbf{a}_\lambda(t_k - \mathbf{v}_{\lambda,\pi})-}(i_k)=1\}} \\
&+ \int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda,\pi,\mathfrak{P}}(i)=0\}} \,\mathrm{d}N_s^S(i) \\
&+ \int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda,\pi,\mathfrak{P}}(i-1)=2, \zeta_{s-}^{\lambda,\pi,\mathfrak{P}}(i)=1\}} \,\mathrm{d}N_s^P(i-1) \\
&+ \int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda,\pi,\mathfrak{P}}(i+1)=2, \zeta_{s-}^{\lambda,\pi,\mathfrak{P}}(i)=1\}} \,\mathrm{d}N_s^P(i+1) \\
&- 2\int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda,\pi,\mathfrak{P}}(i)=2\}} \,\mathrm{d}N_s^P(i).
\end{aligned}
$$

We now explain the behaviour of the process $(\zeta_t^{\lambda,\pi,\mathfrak{P}}(i))_{t\geq 0, i\in\mathbb{Z}}$.

- If $t_0 = 0$, then the process starts from a vacant initial configuration. The match falling on 0 at time $\mathbf{a}_\lambda t_1 \in (0\,, \mathbf{a}_\lambda)$ creates a barrier, see Lemma II.8.2, because $t_1 \in (0\,, 1)$. Then, fires start in i_k alternately on the right and on the left of 0 at times $\mathbf{a}_\lambda(t_k - \mathbf{v}_{\lambda,\pi})$ for all $k = 2, \ldots, K$ and fires spread accross $\mathbb{Z}$ according to the same rules as the $(\lambda, \pi)-$FFP.

- If $t_0 > 1$, the process starts from an occupied initial situation. Nothing happens until a match falls on i_0 at time $\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi})$ and spreads across $\mathbb{Z}$. Next, a match falls on 0 at time $\mathbf{a}_\lambda t_1 \in (\mathbf{a}_\lambda t_0\,, \mathbf{a}_\lambda(t_0 + 1))$. It then creates a barrier, see Lemma II.8.2. Afterwards, matches fall successively in i_k at time $\mathbf{a}_\lambda(t_k - \mathbf{v}_{\lambda,\pi})$ for each $k = 2, \ldots, K$ and fires spread accross $\mathbb{Z}$ according to the same rules as the $(\lambda, \pi)-$FFP.

For all $\gamma > 0$ such that

$$
2\gamma \leq \min_{i=0,\ldots,K-1} (t_{i+1} - t_i, t_i + 1 - t_{i+1}) \vee \min_{i=0,\ldots,K-2} (t_{i+2} - t_i), \tag{II.8.7}
$$

consider the event

$$
\begin{aligned}
\Omega_{\mathfrak{P}}^{S,P}(\lambda, \pi, \gamma) = \{&\forall k \in \{2, \ldots, K\}, \exists j \in [\![-\mathbf{m}_\lambda \,, \mathbf{m}_\lambda]\!], \\
&\forall t \in [t_k + 2\mathbf{v}_{\lambda,\pi} \,, t_k + 1 - \gamma), \zeta^{\lambda,\pi,\mathfrak{P}}_{\mathbf{a}_\lambda t}(j) = 0\}.
\end{aligned}
$$

Lemma II.8.3. *Let $\mathcal{P} = (t_0, \dots, t_K)$ satisfy $(PP1)$ and $\mathcal{I} = (\varepsilon; i_0, i_2, \dots, i_K)$ satisfy $(PP2)$. For each $\lambda \in (0\,,1)$ and each $\pi \geq 1$, consider the process $(\zeta_t^{\lambda,\pi,\mathfrak{P}}(i))_{t\geq 0, i\in\mathbb{Z}}$ defined above.*

If $t_2 - t_1 < t_1 - t_0$, when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$, for all $\gamma > 0$ satisfying (II.8.7), there holds

$$\lim_{\lambda,\pi} \mathbb{P}\left[\Omega_{\mathfrak{P}}^{S,P}(\lambda,\pi,\gamma)\right] = 1.$$

Proof. We define, recall Definition II.4.7,

$$\Omega_{\lambda,\pi}^{P,T,\mathfrak{P}} = \Omega_{\lambda,\pi}^{P,T}(0,t_1) \cap \bigcap_{k=0,2,\dots,K} \Omega_{\lambda,\pi}^{P,T}\left(\frac{i_k}{\mathbf{n}_\lambda}, t_k - \mathbf{v}_{\lambda,\pi}\right).$$

There holds that $\mathbb{P}\left[\Omega_{\lambda,\pi}^{P,T,\mathfrak{P}}\right]$ tends to 1 as $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$, by Lemma II.4.2. We fix some $\gamma > 0$ satisfying (II.8.7). In the whole proof, we work on $\Omega_{\lambda,\pi}^{P,T,\mathfrak{P}}$ and assume that (λ,π) is sufficiently close to the regime $\mathcal{R}(p)$ in such a way that $3\mathbf{v}_{\lambda,\pi} < \gamma$.

For simplicity, we assume that $\varepsilon = -1$, $t_0 = 0$ and that K is even. The other cases are treated similarly (see for example Step 3 in Lemma II.8.2). Fix $\alpha = 1/K$. We define $\mathcal{M} = (0; 0, t_1)$, recall Lemma II.8.2.

Observe that on $\Omega_{\lambda,\pi}^{P,T,\mathfrak{P}}$, a burning tree at time $\mathbf{a}_\lambda t$ necessarily belongs to $[\![i_k + \lfloor \mathbf{a}_\lambda \pi(t - t_k - \varepsilon_\lambda)\rfloor\,, i_k + \lfloor \mathbf{a}_\lambda\pi(t - t_k + \varepsilon_\lambda)\rfloor]\!]$ or to $[\![i_k - \lfloor \mathbf{a}_\lambda\pi(t - t_k + \varepsilon_\lambda)\rfloor\,, i_k - \lfloor \mathbf{a}_\lambda\pi(t - t_k - \varepsilon_\lambda)\rfloor]\!]$, for some $k \in \{0, \dots, K\}$, and is either a front of a fire or has vacant neighbors.

Observe that for all $i \in [\![-\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}\,, -\mathbf{m}_\lambda]\!]$, we have, recall (II.8.1) and (II.8.2),

$$i + \lfloor \mathbf{a}_\lambda \pi(3\mathbf{v}_{\lambda,\pi} - \varepsilon_\lambda)\rfloor \geq \mathbf{m}_\lambda \tag{II.8.8}$$

while for all $i \in [\![\mathbf{m}_\lambda\,, \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}]\!]$, we have

$$i - \lfloor \mathbf{a}_\lambda \pi(3\mathbf{v}_{\lambda,\pi} - \varepsilon_\lambda)\rfloor \leq -\mathbf{m}_\lambda. \tag{II.8.9}$$

First fire. We put $C^P = C^P((\zeta_t^{\lambda,\pi,\mathfrak{P}}(i))_{t\geq 0, i\in\mathbb{Z}}, (0,t_1))$, the destroyed cluster due to the match falling on 0 at time $\mathbf{a}_\lambda t_1$, recall (II.4.14). Since $0 < t_1 < 1$, there holds $C^P \subset [\![-\lfloor \alpha\mathbf{m}_\lambda\rfloor\,, \lfloor \alpha\mathbf{m}_\lambda\rfloor]\!]$ with probability tending to 1 (use Lemma II.8.1-1, space/time stationarity and **Micro**(p) in Subsection II.4.4). Thus the match falling at time $\mathbf{a}_\lambda t_1$ destroys nothing outside $[\![-\lfloor \alpha\mathbf{m}_\lambda\rfloor\,, \lfloor \alpha\mathbf{m}_\lambda\rfloor]\!]$ and there is no more burning tree in $\mathbb{Z}$ at time $\mathbf{a}_\lambda(t_1 + \kappa_{\lambda,\pi}^0)$.

Second fire. Since $t_2 - \mathbf{v}_{\lambda,\pi} > 1$, at least one seed has fallen, during $[0\,, \mathbf{a}_\lambda(t_2 - \mathbf{v}_{\lambda,\pi}))$, on each site of $[\![-\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}\,, -\lfloor\alpha\mathbf{m}_\lambda\rfloor - 1]\!]$ with probability tending to 1 (use Lemma II.8.1-2 and space/time stationarity). Since this zone has not been affected by a fire during the time interval $[0\,, \mathbf{a}_\lambda(t_2 - \mathbf{v}_{\lambda,\pi}))$, this zone is completely occupied at time $\mathbf{a}_\lambda(t_2 - \mathbf{v}_{\lambda,\pi})-$.

Besides, with probability tending to 1, there is (at least) an empty site in $C^P \subset [\![-\lfloor\alpha\mathbf{m}_\lambda\rfloor\,, \lfloor\alpha\mathbf{m}_\lambda\rfloor]\!]$ during the time interval $(\mathbf{a}_\lambda(t_1 + \kappa_{\lambda,\pi}^0)\,, \mathbf{a}_\lambda(t_2 + 2\mathbf{v}_{\lambda,\pi}))$ because $t_2 +$

$2\mathbf{v}_{\lambda,\pi} < t_1 + \Theta^{\lambda,\pi}_{\mathcal{M}}$ with probability tending to 1 (by Lemma II.8.2, $\Theta^{\lambda,\pi}_{\mathcal{M}} \simeq t_1 - t_0 = t_1$ and $t_2 - t_1 < t_1 - t_0 = t_1$ by assumption) and because by definition of $\Theta^{\lambda,\pi}_{\mathcal{M}}$, there is an empty site in $C^P \subset [\![-\lfloor \alpha \mathbf{m}_\lambda \rfloor , \lfloor \alpha \mathbf{m}_\lambda \rfloor]\!]$ during $[\mathbf{a}_\lambda(t_1 + \kappa^0_{\lambda,\pi}) , \mathbf{a}_\lambda(t_1 + \Theta^{\lambda,\pi}_{\mathcal{M}})]$.

Thus, the fire ignited on $i_2 \in [\![-\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi} , -\mathbf{m}_\lambda]\!]$ at time $\mathbf{a}_\lambda(t_2 - \mathbf{v}_{\lambda,\pi})$ burns each site of $[\![-\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi} , -\lfloor \alpha \mathbf{m}_\lambda \rfloor - 1]\!]$ before $\mathbf{a}_\lambda(t_2 + 2\mathbf{v}_{\lambda,\pi})$ and does not affect the zone $[\![\lfloor \alpha \mathbf{m}_\lambda \rfloor + 1 , \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}]\!]$, thanks to (II.8.8) and $\Omega^{P,T}_{\lambda,\pi}(i_2/\mathbf{n}_\lambda, t_2 - \mathbf{v}_{\lambda,\pi})$ (because the right front of the fire 2 reach a vacant site and thus extinguish).

Third fire. All the sites of $[\![\lfloor \alpha \mathbf{m}_\lambda \rfloor , \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}]\!]$ are occupied at time $\mathbf{a}_\lambda(t_3 - \mathbf{v}_{\lambda,\pi})-$ with probability tending to 1 (because on $\Omega^{P,T}_{\lambda,\pi}(0, t_1) \cap \Omega^{P,T}_{\lambda,\pi}(i_2/\mathbf{n}_\lambda, t_2 - \mathbf{v}_{\lambda,\pi})$, they have not been affected by a fire during $[0 , \mathbf{a}_\lambda(t_3 - \mathbf{v}_{\lambda,\pi}))$ and because $t_3 - \mathbf{v}_{\lambda,\pi} > t_2 - \mathbf{v}_{\lambda,\pi} > 1$, see Lemma II.8.1-2.).

Next, the probability that there is a site in $[\![-\lfloor 2\alpha \mathbf{m}_\lambda \rfloor , -\lfloor \alpha \mathbf{m}_\lambda \rfloor]\!]$ where no seed falls during $[\mathbf{a}_\lambda(t_2 - \mathbf{v}_{\lambda,\pi}) , \mathbf{a}_\lambda(t_2 - \gamma + 1)]$ tends to 1 as $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$ (use Lemma II.8.1-1 and space/time stationarity). Thus, since $t_3 - t_2 < 1 - 2\gamma$, with probability tending to 1, there exists a vacant site in $[\![-\lfloor 2\alpha \mathbf{m}_\lambda \rfloor , -\lfloor \alpha \mathbf{m}_\lambda \rfloor]\!]$ during

$$[\mathbf{a}_\lambda(t_2 + 2\mathbf{v}_{\lambda,\pi}) , \mathbf{a}_\lambda(t_2 - \gamma + 1)] \supset [\mathbf{a}_\lambda(t_3 - \mathbf{v}_{\lambda,\pi}) , \mathbf{a}_\lambda(t_3 + 2\mathbf{v}_{\lambda,\pi})]$$

(because each site of $[\![-\lfloor 2\alpha \mathbf{m}_\lambda \rfloor , -\lfloor \alpha \mathbf{m}_\lambda \rfloor]\!]$ has been made vacant by the second fire during $[\mathbf{a}_\lambda(t_2 - \mathbf{v}_{\lambda,\pi}) , \mathbf{a}_\lambda(t_2 + 2\mathbf{v}_{\lambda,\pi})]$).

Thus, the fire ignited on $i_3 \in [\![\mathbf{m}_\lambda , \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}]\!]$ at time $\mathbf{a}_\lambda(t_3 - \mathbf{v}_{\lambda,\pi})$ burns each site of $[\![\lfloor \alpha \mathbf{m}_\lambda \rfloor + 1 , \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}]\!]$ before $\mathbf{a}_\lambda(t_3 + 2\mathbf{v}_{\lambda,\pi})$ and does not affect the zone $[\![-\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi} , -\lfloor \alpha \mathbf{m}_\lambda \rfloor - 1]\!]$ with probability tending to 1, thanks to (II.8.9) and $\Omega^{P,T}_{\lambda,\pi}(i_3/\mathbf{n}_\lambda, t_3 - \mathbf{v}_{\lambda,\pi})$ (because the left front of the fire 3 reach a vacant site and thus extinguish).

Fourth fire. All the sites of $[\![-\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi} , -\lfloor \alpha \mathbf{m}_\lambda \rfloor - 1]\!]$ are occupied at time $\mathbf{a}_\lambda(t_4 - \mathbf{v}_{\lambda,\pi})-$ with probability tending to 1 (because on $\Omega^{P,T}_{\lambda,\pi}(0, t_1) \cap \Omega^{P,T}_{\lambda,\pi}(i_2/\mathbf{n}_\lambda, t_2 - \mathbf{v}_{\lambda,\pi}) \cap \Omega^{P,T}_{\lambda,\pi}(i_3/\mathbf{n}_\lambda, t_3 - \mathbf{v}_{\lambda,\pi})$, they have not been affected by a fire during $(\mathbf{a}_\lambda(t_2 + 2\mathbf{v}_{\lambda,\pi}) , \mathbf{a}_\lambda(t_4 - \mathbf{v}_{\lambda,\pi}))$ and because $t_4 - 3\mathbf{v}_{\lambda,\pi} - t_2 > 1$, see Lemma II.8.1-2 and spae/time stationarity).

The probability that there is a site in $[\![\lfloor \alpha \mathbf{m}_\lambda \rfloor + 1 , \lfloor 2\alpha \mathbf{m}_\lambda \rfloor]\!]$ where no seed falls during $[\mathbf{a}_\lambda(t_3 - \mathbf{v}_{\lambda,\pi}) , \mathbf{a}_\lambda(t_3 - \gamma + 1)]$ tends to 1 as $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$ (use Lemma II.8.1-1 and space/time stationarity). Hence, since $t_4 - t_3 < 1 - 2\gamma$, there is at least one vacant site in $[\![\lfloor \alpha \mathbf{m}_\lambda \rfloor + 1 , \lfloor 2\alpha \mathbf{m}_\lambda \rfloor]\!]$ during

$$[\mathbf{a}_\lambda(t_3 + 2\mathbf{v}_{\lambda,\pi}) , \mathbf{a}_\lambda(t_3 - \gamma + 1)] \supset [\mathbf{a}_\lambda(t_4 - \mathbf{v}_{\lambda,\pi}) , \mathbf{a}_\lambda(t_4 + 2\mathbf{v}_{\lambda,\pi})],$$

with probability tending to 1.

Thus, the fire ignited on $i_4 \in [\![-\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi} , -\mathbf{m}_\lambda]\!]$ at time $\mathbf{a}_\lambda(t_4 - \mathbf{v}_{\lambda,\pi})$ burns each site of $[\![-\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi} , -\lfloor \alpha \mathbf{m}_\lambda \rfloor - 1]\!]$ before $\mathbf{a}_\lambda(t_4 + 2\mathbf{v}_{\lambda,\pi})$ and does not affect the zone $[\![\lfloor \alpha \mathbf{m}_\lambda \rfloor , \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}]\!]$ with probability tending to 1, thanks to (II.8.8) and $\Omega^{P,T}_{\lambda,\pi}(i_4/\mathbf{n}_\lambda, t_4 - \mathbf{v}_{\lambda,\pi})$.

Last fire and conclusion. Iterating the procedure, we see that with a probability tending to 1 as $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$, the zone $[\![-\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}, -\lfloor (K\alpha/2)\mathbf{m}_\lambda \rfloor - 1]\!] = [\![-\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}, -\lfloor \mathbf{m}_\lambda/2 \rfloor - 1]\!]$ is completely occupied at time $\mathbf{a}_\lambda(t_K - \mathbf{v}_{\lambda,\pi})-$ and there is at least one vacant site in $[\![\lfloor (K-1)\alpha/2\mathbf{m}_\lambda \rfloor, \lfloor (K\alpha/2)\mathbf{m}_\lambda \rfloor]\!]$ during the time interval $(\mathbf{a}_\lambda(t_{K-1} + 2\mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(t_{K-1} - \gamma + 1)) \supset (\mathbf{a}_\lambda(t_K - \mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(t_K + 2\mathbf{v}_{\lambda,\pi}))$. Thus, the fire ignited on $i_K \in [\![-\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}, -\mathbf{m}_\lambda]\!]$ at time $\mathbf{a}_\lambda(t_K - \mathbf{v}_{\lambda,\pi})$ destroys each site of the zone $[\![-\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}, -\lfloor \mathbf{m}_\lambda/2 \rfloor - 1]\!]$ before $\mathbf{a}_\lambda(t_K + 2\mathbf{v}_{\lambda,\pi})$ and does not affect the zone $[\![\mathbf{m}_\lambda/2, \mathbf{m}_\lambda]\!]$, thanks to (II.8.8) and $\Omega^{P,T}_{\mathbf{a}_\lambda,\pi}(i_K/\mathbf{n}_\lambda, t_K - \mathbf{v}_{\lambda,\pi})$.

Finally, the probability that there is at least one site in $[\![-\mathbf{m}_\lambda, -\mathbf{m}_\lambda/2]\!]$ with no seed falling during $[\mathbf{a}_\lambda(t_K - \mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(t_K - \gamma + 1)]$ tends to 1 (by Lemma II.8.1-1.). Consequently, the probability that there is a vacant site in $[\![-\mathbf{m}_\lambda, -\mathbf{m}_\lambda/2]\!]$ during $[\mathbf{a}_\lambda(t_K + 2\mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(t_K - \gamma + 1)]$ tends to 1 (because it has been made vacant by the fire K).

All this implies that for all $k \in \{2, \dots, K\}$,, there is $j \in [\![-\mathbf{m}_\lambda, \mathbf{m}_\lambda]\!]$ such that for all $t \in [t_k + 2\mathbf{v}_{\lambda,\pi}, t_k + 1 - \gamma)$ there holds $\zeta^{\lambda,\pi,\mathfrak{P}}_{\mathbf{a}_\lambda t}(j) = 0$, as desired. $\square$

II.8.4. Heart of the proof

II.8.4.1. The coupling

We are going to construct a coupling between the $(\lambda, \pi, A)-$FFP (on the time interval $[0, \mathbf{a}_\lambda T]$) and the $A-$LFFP(p) (on $[0, T]$). Let π_M be a Poisson measure on $\mathbb{R} \times [0, \infty)$ with intensity measure $\mathrm{d}x\,\mathrm{d}t$.

First, we take for the matches of the discrete process the Poisson processes

$$N^M_t(i) = \pi_M([i/\mathbf{n}_\lambda, (i+1)/\mathbf{n}_\lambda) \times [0, t/\mathbf{a}_\lambda])$$

for all $i \in \mathbb{Z}$ and $t \in [0, T]$.

We call $n := \pi_M([0, T] \times [-A, A])$ and we consider the marks $(T_q, X_q)_{q=1,\dots,n}$ of π_M ordered in such a way that $0 < T_1 < \dots < T_n < T$.

Next, we introduce some i.i.d. families of i.i.d. Poisson processes $(N^{S,q}_t(i))_{t\geq 0, i\in\mathbb{Z}}$ and $(N^{P,q}_t(i))_{t\geq 0, i\in\mathbb{Z}}$ with respective parameter 1 and π, for $q = 0, 1, \dots,$ independent of π_M.

Then we build two families of i.i.d. Poisson processes $(N^{S,\lambda,\pi}_t(i))_{t\geq 0, i\in\mathbb{Z}}$ and $(N^{P,\lambda,\pi}_t(i))_{t\geq 0, i\in\mathbb{Z}}$ as follows.

- For $q \in \{1, \dots, n\}$, for all $i \in [X_q]_{\lambda,\pi}$, set $(N^{S,\lambda,\pi}_t(i))_{t\geq 0} = (N^{S,q}_t(i - \lfloor \mathbf{n}_\lambda X_q \rfloor))_{t\geq 0}$ and $(N^{P,\lambda,\pi}_t(i))_{t\geq 0} = (N^{P,q}_t(i - \lfloor \mathbf{n}_\lambda X_q \rfloor))_{t\geq 0}$ (if i belongs to $[X_q]_{\lambda,\pi} \cap [X_r]_{\lambda,\pi}$ for some $q < r$, set *e.g.* $(N^{S,\lambda,\pi}_t(i))_{t\geq 0} = (N^{S,q}_t(i - \lfloor \mathbf{n}_\lambda X_q \rfloor))_{t\geq 0}$ and $(N^{P,\lambda,\pi}_t(i))_{t\geq 0} = (N^{P,q}_t(i - \lfloor \mathbf{n}_\lambda X_q \rfloor))_{t\geq 0}$. This will occur with a very small probability, so that this choice is not important).
- For all other $i \in \mathbb{Z}$ set $(N^{S,\lambda,\pi}_t(i))_{t\geq 0} = (N^{S,0}_t(i))_{t\geq 0}$ and $(N^{P,\lambda,\pi}_t(i))_{t\geq 0} = (N^{P,0}_t(i))_{t\geq 0}$.

The $(\lambda, \pi, A)-$FFP $(\eta^{\lambda,\pi}_t(i))_{t\geq 0, i\in I^\lambda_A}$ is built from the seed processes $(N^{S,\lambda,\pi}_t(i))_{t\geq 0, i\in\mathbb{Z}}$, the match processes $(N^M_t(i))_{t\geq 0, i\in\mathbb{Z}}$ and the propagation processes $(N^{P,\lambda,\pi}_t(i))_{t\geq 0, i\in\mathbb{Z}}$.

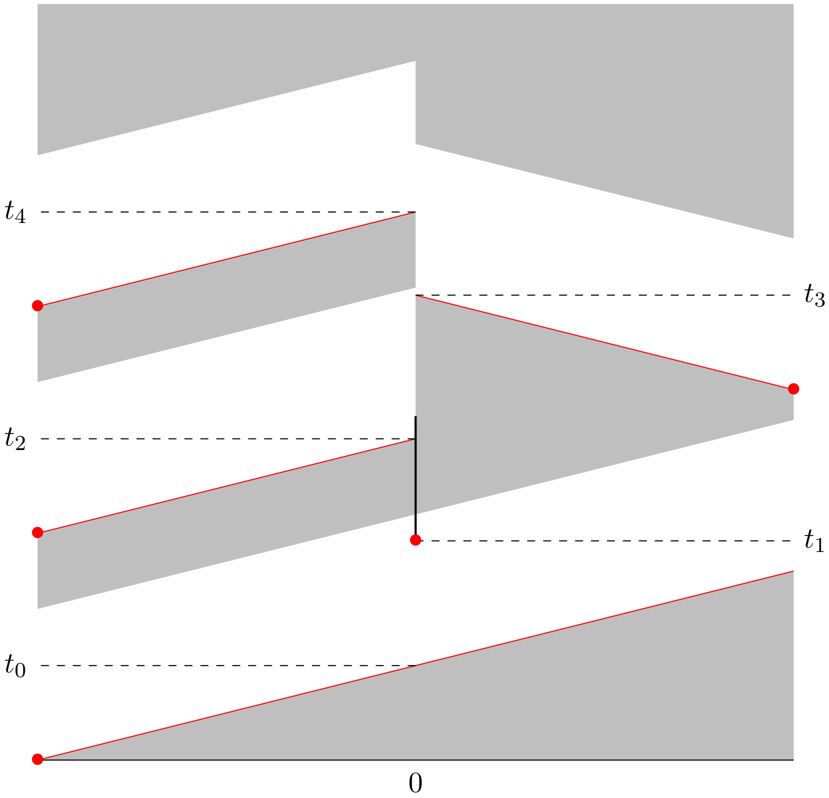

Figure II.7.: Persistent effect of microscopic fires.

Here we focus on the limit process with $t_0 > 1$. A first fire starts at time $\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi})$ and spread across $\mathbb{Z}$. Thus, the match falling in 0 at time $\mathbf{a}_\lambda t_1$ creates a barrier during $\mathbf{a}_\lambda(t_1 - t_0)$. If there are alternatively macroscopic fires on the left and right, there necessarily exists a vacant site around 0 during $(\mathbf{a}_\lambda(t_0 + 2\mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(t_K + 1 - \mathbf{v}_{\lambda,\pi}))$.

Finally, we build the $A-$LFFP(p) $(Z_t(x), H_t(x), F_t(x))_{t\in[0,T],x\in[-A,A]}$ from π_M and observe that it is independent of $(N_t^{S,q}(i))_{t\in[0,\mathbf{a}_\lambda T],i\in\mathbb{Z},q\geq 0}$ and $(N_t^{P,q}(i))_{t\in[0,\mathbf{a}_\lambda T],i\in\mathbb{Z},q\geq 0}$.

Observe that if a match falls at some X_q at time T_q for the LFFP(p), it will fall at $\lfloor \mathbf{n}_\lambda X_q \rfloor$ at time $\mathbf{a}_\lambda T_q$ in the discrete process, and thus if the resulting fire is microscopic in the limit process, it will involve with the same seed and propagation processes for all values of λ and π in discrete process.

II.8.4.2. A favorable event

We set $T_0 = 0$ and introduce

$$\mathcal{T}_M = \{T_0, T_1, \ldots, T_n\} \text{ and } \mathcal{B}_M = \{X_1, \ldots, X_n\}.$$

For $q \in \{1, \dots, n\}$, $x \in [-A\,, A]$ and $t \in [0\,, T]$, we define

$$T_q(x) = T_q + p|x - X_q| \tag{II.8.10}$$

$$X_q^+(t) = X_q + \frac{t - T_q}{p} \tag{II.8.11}$$

$$X_q^-(t) = X_q - \frac{t - T_q}{p} \tag{II.8.12}$$

which are respectively the possible transit time in x of the fire starting in X_q at time T_q and the possible location of the right and the left front at time t of the fire starting in X_q at time T_q. Observe that all $x \in [-A\,, A]$ either equal to $X_k^+(T_k(x))$ or $X_k^-(T_k(x))$.

We next introduce, for $q \in \{1, \dots, n\}$,

$$\mathcal{S}_{M,q} = \{T_k(X_q) = T_k + p\,|X_q - X_k| : k \neq q\}$$

the set of all the possible transit times in X_q of the other fire k and

$$\mathcal{S}_M = \cup_{q=1,\dots,n}\, \mathcal{S}_{M,q}.$$

We also introduce

$$\mathcal{S}_M^1 = \{2T_q - s : q \in \{1, \dots, n\}, s \in \mathcal{S}_{M,q}, s < T_q\}$$

which has to be seen as the set of the possible end of the microscopic fires, recall Lemma II.8.2 and, for $q \in \{2, \dots, n\}$,

$$\mathcal{B}_{M,q}^1 = \left\{X_k^+(T_q) = X_k + \frac{T_q - T_k}{p} : 1 \leq k < q\right\} \cup \left\{X_k^-(T_q) = X_k + \frac{T_k - T_q}{p} : 1 \leq k < q\right\}$$

which has to be seen as the set of the possible locations of the fire k at time T_q.

We finally introduce

$$\mathcal{B}_M^2 = \left\{\frac{T_q - T_k}{2p} + \frac{X_q + X_k}{2} : X_k < X_q\right\} \text{ and } \mathcal{S}_M^2 = \left\{\frac{T_q + T_k}{2} + p\frac{X_q + X_k}{2} : 1 \leq k < q \leq n\right\}$$

which has to be seen as the set of the possible locations and the set of the possible times where two fires may meet as well as the set $\mathcal{C}_M$ of connected component of $[-A\,, A] \setminus (\mathcal{B}_M \cup \mathcal{B}_M^2)$ (sometimes refers as cells).

For $\alpha > 0$, we consider the event

$$\Omega_M(\alpha) = \left\{ \min_{\substack{s,t \in \mathcal{T}_M \cup \mathcal{S}_M \cup \mathcal{S}_M^1 \cup \mathcal{S}_M^2, \\ s \neq t}} |t - s| \geq 3\alpha, \min_{s,t \in \mathcal{T}_M \cup \mathcal{S}_M \cup \mathcal{S}_M^1 \cup \mathcal{S}_M^2,} |t - (s+1)| \geq 3\alpha, \right.$$
$$\left. \min_{\substack{x,y \in \mathcal{B}_M \cup \mathcal{B}_M^2 \cup \{-A, A\}, \\ x \neq y}} |x - y| \geq \frac{3\alpha}{p} \right\}$$

which clearly satisfies $\lim_{\alpha\to 0}\mathbb{P}[\Omega_M(\alpha)] = 1$. For any given $\alpha > 0$, there exists $\lambda_\alpha \in (0\,,1)$ and $\varepsilon_\alpha > 0$ such that for all $\lambda \in (0\,,\lambda_\alpha)$ and all $\pi \geq 1$ in such a way that $|\mathbf{n}_\lambda/(\mathbf{a}_\lambda\pi) - p| < \varepsilon_\alpha$, on $\Omega_M(\alpha)$, there holds that for all $x, y \in \mathcal{B}_M \cup \mathcal{B}_M^2 \cup \{-A, A\}$, with $x \neq y$, $[x]_{\lambda,\pi} \cap [y]_{\lambda,\pi} = \emptyset$.

For $q \in \{1, \ldots, n\}$, using the seed processes $(N_t^{S,\lambda,\pi}(i))_{t\geq 0, i\in\mathbb{Z}}$ and the propagation processes $(N_t^{P,\lambda,\pi}(i))_{t\geq 0,i\in\mathbb{Z}}$, we build, recall Definition II.4.6, $(\zeta_t^{\lambda,\pi,q}(i))_{t\geq 0,i\in\mathbb{Z}}$ (the propagation process ignited at (X_q, T_q)), $(i_t^{q,+})_{t\geq 0}$ and $(i_t^{q,-})_{t\geq 0}$ (the corresponding right and left fronts) and $(T_i^q)_{i\in\mathbb{Z}}$ (the associated burning times). We also use $\Omega^{P,T}_{\lambda,\pi}(X_q, T_q)$, recall Definition II.4.7. We set

$$\Omega^{P,T}(\lambda, \pi) = \bigcap_{q=1,\ldots,n} \Omega^{P,T}_{\lambda,\pi}(X_q, T_q).$$

Since π_M is independent of the processes $(N_t^{S,\lambda,\pi}(i))_{t\geq 0,i\in\mathbb{Z}}$ and $(N_t^{P,\lambda,\pi}(i))_{t\geq 0,i\in\mathbb{Z}}$, Lemma II.4.2 implies that $\mathbb{P}\left[\Omega^{P,T}(\lambda, \pi)\right]$ tends to 1 when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$.

Let $q \in \{1, \ldots, n\}$. We define

$$\mathcal{I}^{q,+} := \left\{\lfloor \mathbf{n}_\lambda X_k\rfloor + i^{k,+}_{\mathbf{a}_\lambda(T_k(X_q)-\mathbf{v}_{\lambda,\pi}-T_k)} - \lfloor \mathbf{n}_\lambda X_k^+(T_k(X_q))\rfloor : k \neq q\right\} \tag{II.8.13}$$

$$\mathcal{I}^{q,-} := \left\{\lfloor \mathbf{n}_\lambda X_k\rfloor + i^{k,-}_{\mathbf{a}_\lambda(T_k(X_q)-\mathbf{v}_{\lambda,\pi}-T_k)} - \lfloor \mathbf{n}_\lambda X_k^-(T_k(X_q))\rfloor : k \neq q\right\}. \tag{II.8.14}$$

Observe that, on $\Omega^{P,T}(\lambda, \pi)$, $\mathcal{I}^{q,-} \subset [\![\mathbf{m}_\lambda\,, \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}]\!]$ while $\mathcal{I}^{q,+} \subset [\![-\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}\,, -\mathbf{m}_\lambda]\!]$. We then call $\mathcal{U}_q$ the set of all possible $\mathfrak{P} = (\mathcal{P}, \mathcal{I})$ satisfying (PP) where

- $\mathcal{P} = (t_0, T_q, t_2, \ldots, t_K)$ satisfies $(PP1)$ with $\{t_0, t_2, \ldots, t_K\} \subset \mathcal{S}_{M,q} \cup \{0\}$ and with $T_q - t_0 > t_2 - T_q$;
- $\mathcal{I} = (\varepsilon; i_0, i_2, \ldots, i_K)$ satisfies $(PP2)$ with $\varepsilon \in \{-1, 1\}$ and $\{i_0, i_2, \ldots, i_K\} \subset \mathcal{I}^{q,+} \cup \mathcal{I}^{q,-}$.

For $\mathfrak{P} \in \mathcal{U}_q$, we introduce the event $\Omega^{S,P,q}_{\mathfrak{P}}(\lambda, \pi, \alpha)$, defined as in Subsection II.8.3, with the Poisson processes $(N_t^{S,q}(i))_{t\geq 0,i\in\mathbb{Z}}$ and $(N_t^{P,q}(i))_{t\geq 0,i\in\mathbb{Z}}$. Then we put

$$\Omega_1^{S,P}(\lambda, \pi, \alpha) = \bigcap_{q=1}^{n} \left\{\text{for all } \mathfrak{P} \in \mathcal{U}_q,\ \Omega^{S,P,q}_{\mathfrak{P}}(\lambda, \pi, \alpha) \text{ holds}\right\},$$

which satisfies $\lim_{\lambda,\pi}\mathbb{P}\left[\Omega_1^{S,P}(\lambda, \pi, \alpha)\right] = 1$ when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$. Indeed, by construction, π_M is independent of $(N_t^{S,q}(i))_{t\geq 0,i\in\mathbb{Z}}$ and $(N_t^{P,q}(i))_{t\geq 0,i\in\mathbb{Z}}$. Observe that for $l \in \{1, \ldots, n\}$, the location $i^{l,+}_{\mathbf{a}_\lambda(T_l(X_q)-\mathbf{v}_{\lambda,\pi}-T_l)}$ depends only on the propagation process $N^{P,\lambda,\pi}$ restricted to $[\mathbf{a}_\lambda T_l\,, \mathbf{a}_\lambda(T_l(X_q) - \mathbf{v}_{\lambda,\pi})] \times \mathbb{Z}$ whereas the event $\Omega^{S,P,q}_{\mathfrak{P}}(\lambda, \pi)$ depends on the location only after $\mathbf{a}_\lambda(T_l(X_q) - \mathbf{v}_{\lambda,\pi})$. Thus, it suffices to work with some fixed $\{t_0, t_2, \ldots, t_K\} \subset \mathcal{S}_{M,q}$ and some fixed $(i_k)_{k=0,2,\ldots,K} \subset \mathcal{I}^{q,+} \cup \mathcal{I}^{q,-}$. The result then follows from Lemma II.8.3.

We also consider the event $\Omega_2^S(\lambda,\pi)$ on which the following conditions hold: for all $t_1, t_2 \in \mathcal{T}_M \cup \mathcal{S}_M \cup \mathcal{S}_M^1$ with $0 < t_2 - t_1 < 1$, for all $q = 1, \dots, n$, there are

$$-\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi} < i_1 < -\mathbf{m}_\lambda < i_2 < 0 < i_3 < \mathbf{m}_\lambda < i_4 < \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}$$

such that $N^{S,q}_{\mathbf{a}_\lambda(t_2+4\mathbf{v}_{\lambda,\pi})}(i_j) - N^{S,q}_{\mathbf{a}_\lambda(t_1-4\mathbf{v}_{\lambda,\pi})}(i_j) = 0$ for $j = 1, \dots, 4$. There holds that $\mathbb{P}\left[\Omega_2^S(\lambda,\pi)\right]$ tends to 1 as $\lambda \to$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$. Indeed, it suffices to prove that almost surely, $\lim_{\substack{\lambda\to 0\\ \pi\to\infty}} \mathbb{P}\left[\Omega_2^S(\lambda,\pi) \,\middle|\, \pi_M\right] = 1$. Since there are a.s. finitely many possibilities for q, t_1, t_2 and since π_M is independent of $(N_t^{S,q}(i))_{t\geq 0, i\in\mathbb{Z}}$, it suffices to work with a fixed $q \in \{1, \dots, n\}$ and some fixed $0 < t_2 - t_1 < 1$. The result then follows from Lemma II.8.1-1,8 together with space/time stationarity and the fact that $\mathbf{v}_{\lambda,\pi} \to 0$.

Next we introduce the event $\Omega_3^S(\lambda,\pi)$ on which the following conditions hold: for all $q \in \{1, \dots, n\}$ and all $i \in I_A^\lambda$

$$N^{S,\lambda,\pi}_{\mathbf{a}_\lambda(T_q(i/\mathbf{n}_\lambda)+1+\mathbf{e}_{\lambda,\pi})}(i) - N^{S,\lambda,\pi}_{\mathbf{a}_\lambda(T_q(i/\mathbf{n}_\lambda)+\mathbf{e}_{\lambda,\pi})}(i) > 0$$

and if $T_q(i/\mathbf{n}_\lambda) \geq 1$,

$$N^{S,\lambda,\pi}_{\mathbf{a}_\lambda(T_q(i/\mathbf{n}_\lambda)-4\mathbf{v}_{\lambda,\pi})}(i) - N^{S,\lambda,\pi}_{\mathbf{a}_\lambda(T_q(i/\mathbf{n}_\lambda)-1-4\mathbf{v}_{\lambda,\pi})}(i) > 0.$$

There holds that $\mathbb{P}\left[\Omega_3^S(\lambda,\pi)\right]$ tends to 1 as $\lambda \to$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$. Observing that $\left|I_A^\lambda\right| \simeq 2A\mathbf{n}_\lambda$, Lemma II.8.1 and space/time stationarity shows the result.

We also need $\Omega_4^{S,P}(\gamma,\lambda,\pi)$, defined for $\gamma > 0$ as follows: for all $q = 1, \dots, n$, for all $\mathcal{M} = (i_0; t_0, T_q)$ such that $t_0 \in \mathcal{S}_{M,q} \cup \{0\}$ with $t_0 < T_q < t_0 + 1$ and $i_0 \in \mathcal{I}^{q,+} \cup \mathcal{I}^{q,-}$, there holds that $\left|\Theta_{\mathcal{M}}^{\lambda,\pi,q} - (T_q - t_0)\right| < \gamma$. Here, $\Theta_{\mathcal{M}}^{\lambda,\pi,q}$ is defined as in Lemma II.8.2 with the seed processes family $(N_t^{S,q}(i))_{t\geq 0, i\in\mathbb{Z}}$ and the propagation processes family $(N_t^{P,q}(i))_{t\geq 0, i\in\mathbb{Z}}$. Lemma II.8.2 directly implies that for any $\gamma > 0$, $\mathbb{P}\left[\Omega_4^{S,P}(\gamma,\lambda,\pi)\right]$ tends to 1 as $\lambda \to$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$.

We finally introduce the event

$$\Omega(\alpha,\gamma,\lambda,\pi) = \Omega_M(\alpha) \cap \Omega^{P,T}(\lambda,\pi) \cap \Omega_1^{S,P}(\lambda,\pi,\alpha) \cap \Omega_2^S(\lambda,\pi) \cap \Omega_3^S(\lambda,\pi) \cap \Omega_4^{S,P}(\gamma,\lambda,\pi).$$

We have shown that for any $\delta > 0$, there exists $\alpha \in (0\,,1)$ such that for any $\gamma > 0$, there holds that $\mathbb{P}\left[\Omega(\alpha,\gamma,\lambda,\pi)\right] > 1 - \delta$ for all (λ,π) sufficiently close to the regime $\mathcal{R}(p)$.

II.8.4.3. Heart of the proof

Consider the $A-$LFFP(p) $(Z_t(x), H_t(x), F_t(x))_{t\geq 0, x\in[-A,A]}$.

For $x \in (-A\,,A)$, we put

$$Z_{t-}(x) = \lim_{s \nearrow t} Z_s(x),$$

$$Z_t(x+) = \lim_{y \searrow x} Z_t(y) \text{ and } Z_t(x-) = \lim_{y \nearrow x} Z_t(y),$$

$$Z_{t-}(x+) = \lim_{y \searrow x} Z_{t-p(y-x)-}(y) \text{ and } Z_{t-}(x-) = \lim_{y \nearrow x} Z_{t+p(y-x)-}(y).$$

For $t \in [0\,,T]$, we set

$$\begin{aligned}
\chi_t^+ &= \{x \in [-A\,,A] : F_t(x) > 0 \text{ and } Z_t(x+) = 1\}\,,\\
\chi_t^- &= \{x \in [-A\,,A] : F_t(x) > 0 \text{ and } Z_t(x-) = 1\}\,,\\
\chi_t^0 &= \{x \in [-A\,,A] : H_t(x) > 0 \text{ or } (F_t(x) = 0 \text{ and } Z_t(x+) \neq Z_t(x-))\} \cup \{-A, A\},\\
\chi_t &= \chi_t^+ \cup \chi_t^- \cup \chi_t^0.
\end{aligned}$$

For $x \in \mathcal{B}_M$ and $t \geq 0$ we set

$$\tilde{H}_t(x) = \max(H_t(x), 1 - Z_t(x), 1 - Z_t(x+), 1 - Z_t(x-)). \tag{II.8.15}$$

Actually, $Z_{t-}(x)$ always equals either $Z_{t-}(x-)$ or $Z_{t-}(x+)$ and these can be distinct only at a point where has occured a microscopic fire (that is if $x = X_q$ for some $q \in \{1\dots, n\}$ with $T_q < t$ and $Z_{T_q-}(X_q) < 1$).

For all $x \in (-A\,,A)$ we define for all $t \in [0\,,T]$

$$\tau_t(x) = \sup\left\{s \leq t : F_s(x) > 0 \text{ and } \tilde{H}_{s-}(x) = 0\right\} \vee 0, \tag{II.8.16}$$

which represents the last time before t that a macroscopic fire has crossed x. Observe that

$$\text{for } x \notin \mathcal{B}_M,\ Z_t(x) = \min(t - \tau_t(x), 1) \text{ for all } t \in [0\,,T], \tag{II.8.17}$$

$$\text{for } q = 1, \dots, n,\ Z_t(X_q) = \min(t - \tau_t(X_q), 1) \text{ for all } t \in [0\,,T_q). \tag{II.8.18}$$

We also define for all $i \in I_A^\lambda$ and all $t \in [0\,,T]$

$$\rho_t^{\lambda,\pi}(i) = \sup\left\{s \leq t : \eta_{\mathbf{a}_\lambda s-}^{\lambda,\pi}(i) = 2\right\} \tag{II.8.19}$$

where $\mathbf{a}_\lambda \rho_t^{\lambda,\pi}(i)$ represents the last time before $\mathbf{a}_\lambda t$ that the site i has been burnt in the discrete process (with the convention $\eta_{0-}^{\lambda,\pi}(i) = 2$ and $\eta_0^{\lambda,\pi}(i) = 0$ for all $i \in I_A^\lambda$).

For $q \in \{1, \dots, n\}$, we define *the death time of the right front of the q's fire* as the time where the fire q is stopped in the limit process, that is,

$$T_q^{D,+} = \inf\left\{t \geq T_q : F_t\left(X_q + \frac{t - T_q}{p}\right) = 0\right\} \tag{II.8.20}$$

as well as *the death position of the right front of the q's fire* as the position where the fire q is stopped in the limit process, that is,

$$X_q^{D,+} = X_q + \frac{T_q^{D,+} - T_q}{p}. \tag{II.8.21}$$

Similarly, *the death time and position of the left front of the q's fire* are defined as

$$T_q^{D,-} = \inf\left\{t \geq T_q : F_t(X_q - \frac{t-T_q}{p}) = 0\right\} \text{ and } X_q^{D,-} = X_q - \frac{T_q^{D,-} - T_q}{p}.$$

Observe that, if $Z_{T_q-}(X_q) < 1$, then $T_q^{D,-} = T_q = T_q^{D,+}$ and $X_q^{D,+} = X_q = X_q^{D,-}$.
We set

$$\mathcal{B}_M^D \coloneqq \{X_1^{D,+}, X_1^{D,-}, \ldots, X_n^{D,+}, X_n^{D,-}\} \subset \mathcal{B}_M \cup \mathcal{B}_M^2, \tag{II.8.22}$$

$$\mathcal{T}_M^D \coloneqq \{T_1^{D,+}, T_1^{D,-}, \ldots, T_n^{D,+}, T_n^{D,-}\} \subset \mathcal{T}_M \cup \mathcal{S}_M \cup \mathcal{S}_M^2. \tag{II.8.23}$$

Let $t \in [0\,, T]$ and $q \in \{1, \ldots, n\}$. If $t \in [0\,, T_q^{D,+} + \mathbf{v}_{\lambda,\pi})$, we set

$$\Omega_{q,t}^{\lambda,\pi,+} = \{\forall s \in [T_q\,, (T_q^{D,+} - \mathbf{v}_{\lambda,\pi}) \wedge t], \eta_{\mathbf{a}_\lambda s}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda X_q \rfloor + i_{\mathbf{a}_\lambda(s-T_q)}^{q,+}) = 2\}$$

and, if $t \in [T_q^{D,+} + \mathbf{v}_{\lambda,\pi}\,, T]$, we set

$$\Omega_{q,t}^{\lambda,\pi,+} = \Omega_{q,T_q^{D,+}}^{\lambda,\pi,+} \cap \{\exists s \in [T_q^{D,+} - \mathbf{v}_{\lambda,\pi}\,, T_q^{D,+} + \mathbf{v}_{\lambda,\pi}], \eta_{\mathbf{a}_\lambda s}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda X_q \rfloor + i_{\mathbf{a}_\lambda(s-T_q)}^{q,+}) = 0\}.$$

Similarly, we set, if $t \in [0\,, T_q^{D,-} + \mathbf{v}_{\lambda,\pi})$,

$$\Omega_{q,t}^{\lambda,\pi,-} = \{\forall s \in [T_q\,, (T_q^{D,-} - \mathbf{v}_{\lambda,\pi}) \wedge t], \eta_{\mathbf{a}_\lambda s}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda X_q \rfloor + i_{\mathbf{a}_\lambda(s-T_q)}^{q,-}) = 2\}$$

and, if $t \in [T_q^{D,-} + \mathbf{v}_{\lambda,\pi}\,, T]$, we set

$$\Omega_{q,t}^{\lambda,\pi,-} = \Omega_{q,T_q^{D,-}}^{\lambda,\pi,-} \cap \{\exists s \in [T_q^{D,-} - \mathbf{v}_{\lambda,\pi}\,, T_q^{D,-} + \mathbf{v}_{\lambda,\pi}], \eta_{\mathbf{a}_\lambda s}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda X_q \rfloor + i_{\mathbf{a}_\lambda(s-T_q)}^{q,-}) = 0\}.$$

Finally, we set, for all $t \in [0\,, T]$,

$$\Omega_t^{\lambda,\pi} = \bigcap_{q=1,\ldots,n} \left(\Omega_{q,t}^{\lambda,\pi,+} \cap \Omega_{q,t}^{\lambda,\pi,-}\right).$$

The aim of this section is to prove the following Lemma.

Lemma II.8.4. *Let $\alpha > \gamma > 0$. For all (λ, π) sufficiently close to the regime $\mathcal{R}(p)$ in such a way that $4(\mathbf{v}_{\lambda,\pi} + p(\mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi})/\mathbf{n}_\lambda) \leq \alpha$, $\Omega_T^{\lambda,\pi}$ a.s. holds on $\Omega(\alpha, \gamma, \lambda, \pi)$.*

We work on $\Omega(\alpha, \gamma, \lambda, \pi)$. We fix $\varepsilon_\alpha > 0$ and $\lambda_\alpha \in (0\,, 1)$ such that for all $\lambda \in (0\,, \lambda_\alpha)$ and all $\pi \geq 1$ in such a way $|\mathbf{n}_\lambda/(\mathbf{a}_\lambda \pi) - p| < \varepsilon_\alpha$, we have $4(\mathbf{v}_{\lambda,\pi} + p(\mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi})/\mathbf{n}_\lambda) \leq \alpha$. Observe that for all $x, y \in \mathcal{B}_M \cup \mathcal{B}_M^2 \cup \{-A, A\}$, with $x \neq y$, we then have $[x]_{\lambda,\pi} \cap [y]_{\lambda,\pi} = \emptyset$. Clearly, $\Omega_{T_1}^{\lambda,\pi}$ a.s. holds, because no match falls in I_A^λ before $\mathbf{a}_\lambda T_1$. We will show that for $q = 0, \ldots, n-1$, $\Omega_{T_q}^{\lambda,\pi}$ implies $\Omega_{T_{q+1}}^{\lambda,\pi}$. This will prove that $\Omega_{T_n}^{\lambda,\pi}$ holds. The extension to $\Omega_T^{\lambda,\pi}$ will be straightforward and will be omitted.

We thus fix $q \in \{0, \dots, n-1\}$ and assume $\Omega^{\lambda,\pi}_{T_q}$. Let $\mathcal{A}_q$ be the set of points where a fire stops during the time interval $(T_q\,, T_{q+1})$ that is, $(x,t) \in \mathcal{A}_q$ if $(x,t) = (X_k^{D,+}, T_k^{D,+})$ (or $(X_k^{D,-}, T_k^{D,-})$) for some $k \le q$ with $T_k^{D,+}$ (or $T_k^{D,+}$) in $(T_q\,, T_{q+1})$. We then put

$$\{(X_q^0, T_q^0), \dots, (X_q^{N_q+1}, T_q^{N_q+1})\} = \mathcal{A}_q \cup \{(X_q, T_q), (X_{q+1}, T_{q+1})\}$$

ordered chronologically (thus $(X_q, T_q) = (X_q^0, T_q^0)$ and $(X_{q+1}, T_{q+1}) = (X_q^{N_q+1}, T_q^{N_q+1})$).

We recall that if $Z_{T_l-}(X_l) = 1$, for some $l \in \{1, \dots, n\}$, on $\Omega_M(\alpha)$, we have by construction,

- $T_l^{D,+} \wedge T_l^{D,-} \ge T_l + 3\alpha$;
- $Z_{T_l-}(y) = 1$ for all $y \in (X_l - 3\alpha/p\,, X_l + 3\alpha/p)$;
- $F_{T_l(y)}(y) = 1$ and $\tilde{H}_{T_l(y)-}(y) = 0$ for all $y \in (X_l^{D,-}\,, X_l^{D,+})$;
- for all $t \in [T_l\,, T_l^{D,+} - 3\alpha]$ and all $y \in (X_l^+(t)\,, X_l^+(t) + 3\alpha/p)$, $\tilde{H}_t(y) = 0$ (similar thing for $X_l^-(t)$);
- for all $t \in [T_l^{D,+} - 3\alpha\,, T_l^{D,+})$ and all $y \in (X_l^+(t)\,, X_l^+(t) + (T_l^{D,+} - t)/p)$, $Z_t(y) = 1$ (similar thing for $X_l^-(t)$).

Recall that on $\Omega_M(\alpha)$, for all $k \in [\![0\,, N_q]\!]$,

$$T_q^{k+1} - T_q^k > 3\alpha.$$

We decompose the proof in four stages.

– *Stage 0.* We deduce, on $\Omega(\alpha, \gamma, \lambda, \pi)$, the last time that a site has been burned.

– *Stage 1.* We prove that on $\Omega(\alpha, \gamma, \lambda, \pi)$, $\Omega^{\lambda,\pi}_{T_q}$ implies $\Omega^{\lambda,\pi}_{T_q + 4\mathbf{v}_{\lambda,\pi}}$.

– *Stage 2.* We prove that on $\Omega(\alpha, \gamma, \lambda, \pi)$, for $0 \le k < N_q$, $\Omega^{\lambda,\pi}_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}$ implies $\Omega^{\lambda,\pi}_{T_q^{k+1} + 4\mathbf{v}_{\lambda,\pi}}$.

– *Stage 3.* We prove that on $\Omega(\alpha, \gamma, \lambda, \pi)$, $\Omega^{\lambda,\pi}_{T_q^{N_q} + 4\mathbf{v}_{\lambda,\pi}}$ implies $\Omega^{\lambda,\pi}_{T_{q+1}}$, which is the goal.

In the whole proof, we repeatedly use the following estimates. For $k \in \{1, \dots, n\}$ and $t \ge T_k$, there holds that, recall (II.8.1), (II.8.2) and (II.8.3),

$$[\![\lfloor \mathbf{n}_\lambda X_k \rfloor + \lfloor \mathbf{a}_\lambda \pi (t - T_k - \varepsilon_\lambda) \rfloor\,, \lfloor \mathbf{n}_\lambda X_k \rfloor + \lfloor \mathbf{a}_\lambda \pi (t - T_k + \varepsilon_\lambda) \rfloor]\!] \subset \langle X_k^+(t) \rangle_{\lambda,\pi} \quad \text{(II.8.24)}$$

which is the possible location of the right front of the fire k at time $\mathbf{a}_\lambda t$, recall Lemma II.4.2,

$$[\![\lfloor \mathbf{n}_\lambda X_k \rfloor + \lfloor \mathbf{a}_\lambda \pi (t - \mathbf{v}_{\lambda,\pi} - T_k - \varepsilon_\lambda) \rfloor\,, \lfloor \mathbf{n}_\lambda X_k \rfloor + \lfloor \mathbf{a}_\lambda \pi (t - \mathbf{v}_{\lambda,\pi} - T_k + \varepsilon_\lambda) \rfloor]\!]$$
$$\subset [\![\lfloor \mathbf{n}_\lambda X_k^+(t) \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}\,, \lfloor \mathbf{n}_\lambda X_k^+(t) \rfloor - \mathbf{m}_\lambda]\!] \quad \text{(II.8.25)}$$

which is the possible location of the right front of the fire k at time $\mathbf{a}_\lambda(t-\mathbf{v}_{\lambda,\pi})$,

$$\begin{aligned}[t]&[\![\lfloor \mathbf{n}_\lambda X_k\rfloor + \lfloor \mathbf{a}_\lambda\pi(t+\mathbf{v}_{\lambda,\pi}-T_k-\varepsilon_\lambda)\rfloor\,,\lfloor \mathbf{n}_\lambda X_k\rfloor + \lfloor \mathbf{a}_\lambda\pi(t+\mathbf{v}_{\lambda,\pi}-T_k+\varepsilon_\lambda)\rfloor]\!]\\ &\qquad\qquad\subset [\![\lfloor \mathbf{n}_\lambda X_k^+(t)\rfloor + \mathbf{m}_\lambda\,, \lfloor \mathbf{n}_\lambda X_k^+(t)\rfloor + \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}]\!]\end{aligned} \tag{II.8.26}$$

which is the possible location of the right front of the fire k at time $\mathbf{a}_\lambda(t+\mathbf{v}_{\lambda,\pi})$.

For $k\in\{1,\dots,n\}$ and $t\geq T_k$ there also holds true that

$$\lfloor \mathbf{n}_\lambda X_k\rfloor + \lfloor \mathbf{a}_\lambda\pi(t-\mathbf{e}_{\lambda,\pi}-T_k+\varepsilon_\lambda)\rfloor \leq \lfloor \mathbf{n}_\lambda X_k^+(t)\rfloor \tag{II.8.27}$$

and

$$\lfloor \mathbf{n}_\lambda X_k\rfloor + \lfloor \mathbf{a}_\lambda\pi(t-4\mathbf{v}_{\lambda,\pi}-T_k+\varepsilon_\lambda)\rfloor \leq \lfloor \mathbf{n}_\lambda X_k^+(t)\rfloor - \mathbf{m}_\lambda - 3\mathbf{k}_{\lambda,\pi}, \tag{II.8.28}$$

$$\lfloor \mathbf{n}_\lambda X_k\rfloor + \lfloor \mathbf{a}_\lambda\pi(t+4\mathbf{v}_{\lambda,\pi}-T_k-\varepsilon_\lambda)\rfloor \geq \lfloor \mathbf{n}_\lambda X_k^+(t)\rfloor + \mathbf{m}_\lambda + 3\mathbf{k}_{\lambda,\pi}. \tag{II.8.29}$$

Very similar estimations of course hold for $X_k^-(t)$.

Finally, for all $i\in I_A^\lambda$ and all $k\in\{1,\dots,n\}$, there holds that

$$\left[T_k+\frac{|i-\lfloor \mathbf{n}_\lambda x\rfloor|}{\mathbf{a}_\lambda\pi}-\varepsilon_\lambda,\, T_k+\frac{|i-\lfloor \mathbf{n}_\lambda x\rfloor|}{\mathbf{a}_\lambda\pi}+\varepsilon_\lambda\right]\subset\left[T_k\left(\frac{i}{\mathbf{n}_\lambda}\right)-\mathbf{e}_{\lambda,\pi},\, T_k\left(\frac{i}{\mathbf{n}_\lambda}\right)+\mathbf{e}_{\lambda,\pi}\right] \tag{II.8.30}$$

which has to be seen as the time interval where a tree may be burn due to the fire k.

STAGE 0.

In this Stage we fix some $s_0\in[0\,,T]$ and work on $\Omega(\alpha,\gamma,\lambda,\pi)\cap\Omega_{s_0}^{\lambda,\pi}$. We deduce an estimate of the last time that a given site has been burned.

Lemma II.8.5. *Let $s_0\in[0\,,T]$ and q_0 such that $s_0\in[T_{q_0}\,,T_{q_0+1})$. On $\Omega(\alpha,\gamma,\lambda,\pi)\cap\Omega_{s_0}^{\lambda,\pi}$, for all $(i,t)\in I_A^\lambda\times[0\,,s_0]$ such that*

$$i\notin\bigcup_{x\in\chi_t}\langle x\rangle_{\lambda,\pi}\cup\bigcup_{1\leq k\leq q_0}\left([X_k^{D,+}]_{\lambda,\pi}\cup[X_k^{D,-}]_{\lambda,\pi}\right), \tag{II.8.31}$$

then

1. *$\tau_t(i/\mathbf{n}_\lambda)=0$ if and only if $\rho_t^{\lambda,\pi}(i)=0$;*
2. *if $\tau_t(i/\mathbf{n}_\lambda)=T_k(i/\mathbf{n}_\lambda)$, for some $k\in\{1,\dots,q_0\}$, then*

$$\rho_t^{\lambda,\pi}(i)\in\left[T_k+\frac{|i-\lfloor \mathbf{n}_\lambda X_k\rfloor|}{\mathbf{a}_\lambda\pi}-\varepsilon_\lambda,\, T_k+\frac{|i-\lfloor \mathbf{n}_\lambda X_k\rfloor|}{\mathbf{a}_\lambda\pi}+\varepsilon_\lambda\right].$$

The condition (II.8.31) means that, at time t, the site i is neither *near* a burning tree nor *near* a place where a fire has been stopped.

Observe that for (i,t) be as in the statement, in the two cases, there holds that, using (II.8.30),

$$\left|\rho_t^{\lambda,\pi}(i)-\tau_t(i/\mathbf{n}_\lambda)\right|\leq\mathbf{e}_{\lambda,\pi}.$$

Let now $t \in [0\,, s_0]$ and $x \in (-A\,, A)$ in such a way that $[x]_{\lambda,\pi} \cap [y]_{\lambda,\pi} = \emptyset$ for all $y \in \chi_t \cup \mathcal{B}_M^D$. If $\tau_t(x) = T_l(x)$, for some $l \in \{1, \dots, n\}$, then by construction $\tau_t(i/\mathbf{n}_\lambda) = T_l(i/\mathbf{n}_\lambda)$ for all $i \in [x]_{\lambda,\pi}$. Thus, using (II.8.25) and (II.8.26), Lemma II.8.5 implies that for all $i \in (x)_\lambda$,

$$\left|\rho_t^{\lambda,\pi}(i) - \tau_t(x)\right| \leq \mathbf{v}_{\lambda,\pi}$$

whence, using (II.8.28) and (II.8.29), for all $i \in [x]_{\lambda,\pi}$, there holds that

$$\left|\rho_t^{\lambda,\pi}(i) - \tau_t(x)\right| \leq 4\mathbf{v}_{\lambda,\pi}.$$

Proof. Let $s_0 \in [0\,, T]$ and q_0 such that $s_0 \in [T_{q_0}\,, T_{q_0+1})$.

Step 1. The key of the proof is the observation that if a site $i \in I_A^\lambda$ is burning at time $\mathbf{a}_\lambda t \leq \mathbf{a}_\lambda s_0$ then there exists $k \in \{1, \dots, q_0\}$ such that $\zeta_{\mathbf{a}_\lambda(t-T_k)}^{\lambda,\pi,k}(i - \lfloor \mathbf{n}_\lambda X_k \rfloor) = 2$: a burning tree in the $(\lambda, \pi, A)-$FFP corresponds to a burning tree in some propagation process.

Indeed, assume that a match falls on $\lfloor \mathbf{n}_\lambda X_k \rfloor$ at time $\mathbf{a}_\lambda T_k \leq \mathbf{a}_\lambda t$. Recall that the propagation process ignited at (X_k, T_k) is defined using the seed processes $(N_t^{S,\lambda,\pi}(i))_{t\geq 0, i\in\mathbb{Z}}$ and the propagation processes $(N_t^{P,\lambda,\pi}(i))_{t\geq 0, i\in\mathbb{Z}}$. Thus, with our coupling, the right front of the fire in the propagation process $(\zeta_t^{\lambda,\pi,k}(i))_{t\geq 0, i\in\mathbb{Z}}$ at some time $\mathbf{a}_\lambda s$ is $i_{\mathbf{a}_\lambda s}^{k,+}$ whence the (hypothetical) right front of the $(\lambda, \pi, A)-$FFP at time $\mathbf{a}_\lambda(s + T_k)$ is $\lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda s}^{k,+}$. Recall that a spark in the propagation process $(\zeta_t^{\lambda,\pi,k}(i))_{t\geq 0, i\in\mathbb{Z}}$ corresponds to a site $i \in \mathbb{Z}$ where a seed has fallen between the instant at which i propagates for the first time and the instant at which $i + 1$ if $i \geq 0$, or $i - 1$ if $i \leq 0$, propagates for the first time. On $\Omega_{\lambda,\pi}^{P,T}(X_k, T_k)$, such a spark has vacant neighbors. Thus, with our coupling, the site $\lfloor \mathbf{n}_\lambda X_k \rfloor + i$ is a spark in the $(\lambda, \pi, A)-$FFP (that is a burning tree which is not a front of a fire) if the site i is also a spark in the propagation process. Such a spark in the $(\lambda, \pi, A)-$FFP has inevitably vacant neighbors.

Step 2. By Step 1, Lemma II.4.2 and (II.8.24), we deduce that a burning tree at time $\mathbf{a}_\lambda t$ in the $(\lambda, \pi, A)-$FFP necessarily belongs to

$$[\![\lfloor \mathbf{n}_\lambda X_k \rfloor + \lfloor \mathbf{a}_\lambda \pi(t - T_k - \varepsilon_\lambda) \rfloor\,, \lfloor \mathbf{n}_\lambda X_k \rfloor + \lfloor \mathbf{a}_\lambda \pi(t - T_k + \varepsilon_\lambda) \rfloor]\!] \subset \langle X_k^+(t) \rangle_{\lambda,\pi}$$

or to

$$[\![\lfloor \mathbf{n}_\lambda X_k \rfloor - \lfloor \mathbf{a}_\lambda \pi(t - T_k + \varepsilon_\lambda) \rfloor\,, \lfloor \mathbf{n}_\lambda X_k \rfloor - \lfloor \mathbf{a}_\lambda \pi(t - T_k - \varepsilon_\lambda) \rfloor]\!] \subset \langle X_k^-(t) \rangle_{\lambda,\pi}$$

for some $k \in \{1, \dots, q_0\}$ such that $T_k \leq t$.

Conversely, if a site $i \in I_A^\lambda$ is burning at time $\mathbf{a}_\lambda t \leq \mathbf{a}_\lambda s_0$ then there is $k \in \{1, \dots, n\}$ such that, recalling (II.8.30),

$$t \in \left[T_k + \frac{|i - \lfloor \mathbf{n}_\lambda X_k \rfloor|}{\mathbf{a}_\lambda \pi} - \varepsilon_\lambda,\, T_k + \frac{|i - \lfloor \mathbf{n}_\lambda X_k \rfloor|}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda\right] \subset \left(T_k\left(\frac{i}{\mathbf{n}_\lambda}\right) - \mathbf{e}_{\lambda,\pi},\, T_k\left(\frac{i}{\mathbf{n}_\lambda}\right) + \mathbf{e}_{\lambda,\pi}\right).$$

Step 3. Next, we observe that if a site j is burning at some time $\mathbf{a}_\lambda u \le \mathbf{a}_\lambda s_0$, then there is $k \in \{1,\dots,q_0\}$ such that $u \in [T_k + (T^k_{j-\lfloor \mathbf{n}_\lambda X_k \rfloor}/\mathbf{a}_\lambda)\,, T_k + \frac{\lfloor j - \lfloor \mathbf{n}_\lambda X_k \rfloor \rfloor}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda]$ and for all $s \in [T_k\,, T_k + (T^k_{j-\lfloor \mathbf{n}_\lambda X_k \rfloor}/\mathbf{a}_\lambda)]$ we have

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(\lfloor \mathbf{n}_\lambda X_k \rfloor + i^{k,+}_{\mathbf{a}_\lambda(s-T_k)}) = 2$$

if $j \ge \lfloor \mathbf{n}_\lambda X_k \rfloor$ while if $j \le \lfloor \mathbf{n}_\lambda X_k \rfloor$, we have

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(\lfloor \mathbf{n}_\lambda X_k \rfloor + i^{k,-}_{\mathbf{a}_\lambda(s-T_k)}) = 2.$$

Indeed, by construction, a fire starting on $\lfloor \mathbf{n}_\lambda X_k \rfloor$ at time $\mathbf{a}_\lambda T_k$, for some $k \in \{1,\dots,q_0\}$, does not affect the site j before $\mathbf{a}_\lambda T_k + T^k_{j-\lfloor \mathbf{n}_\lambda X_k \rfloor}$ and by $\Omega^{P,T}_{\lambda,\pi}(X_k, T_k)$, as been checked on Step 1, does not affect the site j after $\mathbf{a}_\lambda T_k + \frac{\lfloor j - \lfloor \mathbf{n}_\lambda X_k \rfloor \rfloor}{\pi} + \mathbf{a}_\lambda \varepsilon_\lambda$.

Assume *e.g.* that $j \ge \lfloor \mathbf{n}_\lambda X_k \rfloor$ and that there is $s \in [T_k\,, T_k + (T^k_{j-\lfloor \mathbf{n}_\lambda X_k \rfloor}/\mathbf{a}_\lambda))$ such that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(\lfloor \mathbf{n}_\lambda X_k \rfloor + i^{k,+}_{\mathbf{a}_\lambda(s-T_k)}) = 0$: the right front reaches a vacant site. Since sparks has vacant neighbors, the right front can not propagate more and is stopped (after a while, thanks to our coupling). Hence, the right front cannot reach j.

Step 4. Here we prove that for i and t be as in the statement and if $\tau_t(i/\mathbf{n}_\lambda) = T_k(i/\mathbf{n}_\lambda) > 0$, for some $k \in \{1,\dots,q_0\}$, then i is not affected (in the discrete process) by any fire during the time interval $[\mathbf{a}_\lambda(T_k + \frac{\lfloor i - \lfloor \mathbf{n}_\lambda X_k \rfloor \rfloor}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda)\,, \mathbf{a}_\lambda t]$.

Assume *e.g.* that $i/\mathbf{n}_\lambda = X^+_k(T_k(i/\mathbf{n}_\lambda)) \in \chi^+_{T_k(i/\mathbf{n}_\lambda)}$ and let $l \ne k$ such that $T_l < t$.

If $i/\mathbf{n}_\lambda = X^+_l(T_l(i/\mathbf{n}_\lambda))$,

(a) either $t < T^{D,+}_l$ whence $X^+_l(t) \in \chi^+_t$. Since $X^+_l(t) < i/\mathbf{n}_\lambda$ (because $\tau_t(i/\mathbf{n}_\lambda) = T_k(i/\mathbf{n}_\lambda)$), we necessarily have $\lfloor \mathbf{n}_\lambda X^+_l(t) \rfloor + \mathbf{k}_{\lambda,\pi} \le i$ (because $i \notin \langle X^+_l(t) \rangle_{\lambda,\pi}$). By Step 2, we easily deduce that the right front does not affect the site i during the considered time interval;

(b) or $t \ge T^{D,+}_l$ whence $i/\mathbf{n}_\lambda \ge X^{D,+}_l$. Since $i \notin [X^{D,+}_l]_{\lambda,\pi}$, we deduce that $i \ge \lfloor \mathbf{n}_\lambda X^{D,+}_l \rfloor + \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}$. By $\Omega^{\lambda,\pi}_t$ and (II.8.26), we deduce that there is a site $j \in [X^{D,+}_l]_{\lambda,\pi}$ such that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_l + T^l_{j-\lfloor \mathbf{n}_\lambda X_l \rfloor}}(j) = 0$. By Step 3, we deduce again that the right front does not affect the site i during the considered time interval.

If $i/\mathbf{n}_\lambda = X^-_l(T_l(i/\mathbf{n}_\lambda))$, similar arguments lead to the same conclusion.

Step 5. Here we prove that for i and t be as in the statement, if $\tau_t(i/\mathbf{n}_\lambda) = T_k(i/\mathbf{n}_\lambda) > 0$ for some $k \in \{1,\dots,n\}$, then $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_k + T^k_{i-\lfloor \mathbf{n}_\lambda X_k \rfloor}}(i) = 2$.

Indeed, assume for example that $i/\mathbf{n}_\lambda = X^+_k(T_k(i/\mathbf{n}_\lambda))$, for some $k \in \{1,\dots,n\}$. By construction, there holds that $i/\mathbf{n}_\lambda \le X^{D,+}_k$ and $i/\mathbf{n}_\lambda \le X^+_k(s_0)$ whence $\lfloor \mathbf{n}_\lambda X_k \rfloor \le i \le \lfloor \mathbf{n}_\lambda X^{D,+}_k \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}$ (because $i \notin [X^{D,+}_k]_{\lambda,\pi}$) and $\lfloor \mathbf{n}_\lambda X_k \rfloor \le i \le \lfloor \mathbf{n}_\lambda X^+_k(s_0) \rfloor - \mathbf{k}_{\lambda,\pi}$ (because if $s_0 \le T^{D,+}_k$ then $i \notin \langle X^+_k(s_0) \rangle_{\lambda,\pi}$ and if $s_0 > T^{D,+}_k$ then $\lfloor \mathbf{n}_\lambda X^+_k(s_0) \rfloor - \mathbf{k}_{\lambda,\pi} \ge \lfloor \mathbf{n}_\lambda X^{D,+}_k \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}$). We distinguish two cases.

- If $s_0 \geq T_k^{D,+} - \mathbf{v}_{\lambda,\pi}$, then by $\Omega_{s_0}^{\lambda,\pi}$, we deduce that $\eta_{\mathbf{a}_\lambda s}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda(s-T_k)}^{k,+}) = 2$ for all $s \in [T_k\,, T_k^{D,+} - \mathbf{v}_{\lambda,\pi}]$. This also implies, thanks to (II.8.25), that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_k + T^k_{j-\lfloor \mathbf{n}_\lambda X_k \rfloor}}(j) = 2$ for all $j \in [\![\lfloor \mathbf{n}_\lambda X_k \rfloor\,, \lfloor \mathbf{n}_\lambda X_k^{D,+} \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}]\!]$. It especially holds for i, thanks to the previous observation.
- If $s_0 < T_k^{D,+} - \mathbf{v}_{\lambda,\pi}$, we deduce, by $\Omega^{P,T}(\lambda,\pi)$, (II.8.24) and the previous observation, that

$$\begin{aligned}\lfloor \mathbf{n}_\lambda X_k \rfloor \leq i \leq \lfloor \mathbf{n}_\lambda X_k^+(s_0) \rfloor - \mathbf{k}_{\lambda,\pi} \\ \leq \lfloor \mathbf{n}_\lambda X_k \rfloor + \lfloor \mathbf{a}_\lambda \pi (s_0 - T_k - \varepsilon_\lambda) \rfloor \leq \lfloor \mathbf{n}_\lambda X_k \rfloor + i^{k,+}_{\mathbf{a}_\lambda(s_0 - T_k)}.\end{aligned} \quad \text{(II.8.32)}$$

  Finally, by $\Omega_{s_0}^{\lambda,\pi}$, we have $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda u}(\lfloor \mathbf{n}_\lambda X_k \rfloor + i^{k,+}_{\mathbf{a}_\lambda(u-T_k)}) = 2$ for all $u \in [T_k\,, s_0]$ which implies the claim.

Step 6. We now conclude in the case $\tau_t(i/\mathbf{n}_\lambda) = T_k(i/\mathbf{n}_\lambda) > 0$. By Step 4, we deduce that

$$\rho_t^{\lambda,\pi}(i) \leq T_k + \frac{|i - \lfloor \mathbf{n}_\lambda X_k \rfloor|}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda.$$

By Step 5, we deduce that $\rho_t^{\lambda,\pi}(i) \geq T_k + T^k_{i - \lfloor \mathbf{n}_\lambda X_k \rfloor|}/\mathbf{a}_\lambda$ and conclude using $\Omega^{P,T}(\lambda,\pi)$ that

$$\rho_t^{\lambda,\pi}(i) \geq T_k + \frac{|i - \lfloor \mathbf{n}_\lambda X_k \rfloor|}{\mathbf{a}_\lambda \pi} - \varepsilon_\lambda.$$

Step 7. Finally, if $\tau_t(i/\mathbf{n}_\lambda) = 0$, we conclude, using similar argument as in Step 4 (recall that $i \notin \bigcup_{1 \leq k \leq q_0} \left([X_k^{D,+}]_{\lambda,\pi} \cup [X_k^{D,-}]_{\lambda,\pi} \right)$), that no fire can affect the site i until $\mathbf{a}_\lambda t$ and thus $\rho_t^{\lambda,\pi}(i) = 0$.

Conversely, if $\rho_t^{\lambda,\pi}(i) = 0$, then for all $l \in \{1, \dots, n\}$ such that $T_l(i/\mathbf{n}_\lambda) < t$, we necessarily have $F_{T_l(i/\mathbf{n}_\lambda)}(i/\mathbf{n}_\lambda) = 0$ (else, applying $\Omega_t^{\lambda,\pi}$, one should have $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_l + T^l_{i - \lfloor \mathbf{n}_\lambda X_l \rfloor}}(i) = 2$). This concludes the proof. $\square$

STAGE 1.

The aim of this stage is to prove that on $\Omega(\alpha,\gamma,\lambda,\pi)$, $\Omega_{T_q}^{\lambda,\pi}$ implies $\Omega^{\lambda,\pi}_{T_q + 4\mathbf{v}_{\lambda,\pi}}$.

Observe that for all $i \in I_A^\lambda \setminus \{\lfloor \mathbf{n}_\lambda X_q \rfloor\}$,

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_q}(i) = \eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_q -}(i)$$

while

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_q}(\lfloor \mathbf{n}_\lambda X_q \rfloor) = 2 \mathbf{1}_{\{\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_q -}(\lfloor \mathbf{n}_\lambda X_q \rfloor) = 1\}}.$$

First, we situate the burning trees at time $\mathbf{a}_\lambda T_q$ for the $(\lambda,\pi,A)-$FFP.

Lemma II.8.6. *We work on $\Omega_{T_q}^{\lambda,\pi} \cap \Omega(\alpha,\gamma,\lambda,\pi)$.*

1. *At time $\mathbf{a}_\lambda T_q$, a burning tree which is not located at $\lfloor \mathbf{n}_\lambda X_q \rfloor$ necessarily belongs to $\langle x \rangle_{\lambda,\pi}$, for some $x \in \chi^+_{T_q} \cup \chi^-_{T_q} \subset \mathcal{B}^1_{M,q}$, and is either at $\lfloor \mathbf{n}_\lambda X_k \rfloor + i^{k,+}_{\mathbf{a}_\lambda(T_q-T_k)}$ or at $\lfloor \mathbf{n}_\lambda X_k \rfloor + i^{k,-}_{\mathbf{a}_\lambda(T_q-T_k)}$, for some $k < q$, or has vacant neighbors.*

2. *If $X^+_k(T_q) = X_k + \frac{T_q-T_k}{p} \in \chi^+_{T_q}$ for some $k < q$, then $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_q}(\lfloor \mathbf{n}_\lambda X_k \rfloor + i^{k,+}_{\mathbf{a}_\lambda(T_q-T_k)}) = 2$ and $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_q}(i) = 1$ for all $i \in [\![\lfloor \mathbf{n}_\lambda X_k \rfloor + i^{k,+}_{\mathbf{a}_\lambda(T_q-T_k)} + 1 \,, \lfloor \mathbf{n}_\lambda (X_k + 2\alpha/p) \rfloor]\!]$.*

3. *If $X^-_k(T_q) = X_k - \frac{T_q-T_k}{p} \in \chi^-_{T_q}$ for some $k < q$, then $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_q}(\lfloor \mathbf{n}_\lambda X_k \rfloor + i^{k,-}_{\mathbf{a}_\lambda(T_q-T_k)}) = 2$ and $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_q}(i) = 1$ for all $i \in [\![\lfloor \mathbf{n}_\lambda (X_k - 2\alpha/p) \rfloor \,, \lfloor \mathbf{n}_\lambda X_k \rfloor + i^{k,+}_{\mathbf{a}_\lambda(T_q-T_k)} - 1]\!]$.*

Proof. First, observe that, by $\Omega_M(\alpha)$, $|x-y| > 3\alpha/p$ for all $x, y \in \mathcal{B}^1_{M,q} \cup \mathcal{B}^D_M$ with $x \neq y$. Hence, for all $x \in \mathcal{B}^1_{M,q}$, there is a unique $k < q$ such that $x = X^+_k(T_q)$ or $x = X^-_k(T_q)$.

In the whole proof, we work on $\Omega(\alpha, \gamma, \lambda, \pi) \cap \Omega^{\lambda,\pi}_{T_q}$.

Step 1. We first prove 1. As claimed in Step 2 in the proof of Lemma II.8.5, due to $\Omega^{P,T}(\lambda, \pi)$, if a tree burns at time $\mathbf{a}_\lambda T_q$ in the $(\lambda, \pi, A)-$FFP, it necessarily belongs to $\langle X^+_k(T_q) \rangle_{\lambda,\pi}$ or $\langle X^-_k(T_q) \rangle_{\lambda,\pi}$ for some $k < q$ and is either $\lfloor \mathbf{n}_\lambda X_k \rfloor + i^{k,+}_{\mathbf{a}_\lambda(T_q-T_k)}$ or $\lfloor \mathbf{n}_\lambda X_k \rfloor + i^{k,-}_{\mathbf{a}_\lambda(T_q-T_k)}$, or has vacant neighbors.

It remains to prove that if $x \in \mathcal{B}^1_{M,q} \setminus (\chi^+_{T_q} \cup \chi^-_{T_q})$, then there is no burning tree in $\langle x \rangle_{\lambda,\pi}$ at time $\mathbf{a}_\lambda T_q$. We assume *e.g.* that $x = X^+_k(T_q)$ for some $k < q$. Since $x \notin \chi^+_{T_q}$, there holds that $T^{D,+}_k \leq T_q$ whence $T^{D,+}_k \leq T_q - 3\alpha$ and $x \geq X^{D,+}_k + 3\alpha/p$, due to $\Omega_M(\alpha)$. We deduce, by $\Omega^{\lambda,\pi}_{T_q}$, that there is $s \in [T^{D,+}_k - \mathbf{v}_{\lambda,\pi} \,, T^{D,+}_k + \mathbf{v}_{\lambda,\pi}]$ such that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(\lfloor \mathbf{n}_\lambda X_k \rfloor + i^{k,+}_{\mathbf{a}_\lambda(s-T_k)}) = 0$ whence as usual (using (II.8.25) and (II.8.26)) that there is $j \in [X^{D,+}_k]_{\lambda,\pi}$ such that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_k + T^k_{j-\lfloor \mathbf{n}_\lambda X_k \rfloor}}(j) = 0$. Since k is unique, we conclude, using similar arguments as in Step 3 in the proof of Lemma II.8.5, that there can not be burning tree in $\langle x \rangle_{\lambda,\pi}$ at time $\mathbf{a}_\lambda T_q$ (because the right front has been stopped in $[X^{D,+}_k]_{\lambda,\pi}$ and $\lfloor \mathbf{n}_\lambda x \rfloor - \mathbf{k}_{\lambda,\pi} \geq \lfloor X^{D,+}_k \rfloor + \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}$).

Step 2. We next prove 2. Let $k < q$. We set $x := X^+_k(T_q) \in \mathcal{B}^1_{M,q}$. Since $x \notin \mathcal{B}^D_m$, we have $T^{D,+}_k > T_q > T_k$ whence, by $\Omega_M(\alpha)$, $T^{D,+}_k > T_q + 3\alpha > T_k + 6\alpha$. Since $Z_{T_q-}(x) = 1$, there holds that $T_q - \tau_{T_q-}(x) \geq 1$ whence $T_q - \tau_{T_q-}(x) \geq 1 + 3\alpha$, thanks to $\Omega_M(\alpha)$. We deduce that $Z_{T_q-}(y) = 1$ and $T_q - \tau_{T_q-}(y) \geq 1 + \alpha$ for all $y \in [x \,, x + 2\alpha/p]$. We set $\tau_{T_q-}(x) = T_l(x)$, for some $l \in \{0, \ldots, q-1\}$.

Let us fix $i \in [\![\lfloor \mathbf{n}_\lambda x \rfloor + \mathbf{k}_{\lambda,\pi} + 1 \,, \lfloor \mathbf{n}_\lambda (x + 2\alpha/p) \rfloor]\!]$. Observing that $i \notin \bigcup_{x \in \chi_{T_q}} \langle x \rangle_{\lambda,\pi} \cup \bigcup_{1 \leq k \leq q} \left([X^{D,+}_k]_{\lambda,\pi} \cup [X^{D,-}_k]_{\lambda,\pi} \right)$, we deduce from Lemma II.8.5 and by (II.8.30) that $\rho^{\lambda,\pi}_{T_q-}(i) \leq \tau_{T_q-}(i/\mathbf{n}_\lambda) + \mathbf{e}_{\lambda,\pi}$ whence

$$\rho^{\lambda,\pi}_{T_q-}(i) \leq T_q - 1 - \alpha + \mathbf{e}_{\lambda,\pi}.$$

We conclude using $\Omega_3^S(\lambda,\pi)$ that i is occupied at time $\mathbf{a}_\lambda T_q$.

Let now $i \in [\![\lfloor \mathbf{n}_\lambda X_k \rfloor + i^{k,+}_{\mathbf{a}_\lambda(T_q-T_k)} + 1\,, \lfloor \mathbf{n}_\lambda x\rfloor + \mathbf{k}_{\lambda,\pi}]\!]$. The site i has not (yet) been affected by the fire k. By $\Omega_3^S(\lambda,\pi)$, if $\rho^{\lambda,\pi}_{T_q-}(i) = 0$ then i is occupied at time $\mathbf{a}_\lambda T_q$, because $T_q \geq 1$. If $\rho^{\lambda,\pi}_{T_q}(i) > 0$, by $\Omega^{P,T}(\lambda,\pi)$, we necessarily have $\rho^{\lambda,\pi}_{T_q}(i) \in [T_l + \frac{|i-\lfloor \mathbf{n}_\lambda X_l\rfloor|}{\mathbf{a}_\lambda \pi} - \varepsilon_\lambda\,, T_l + \frac{|i-\lfloor \mathbf{n}_\lambda X_l\rfloor|}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda]$. We deduce as above that

$$\rho^{\lambda,\pi}_{T_q}(i) \leq T_l(i/\mathbf{n}_\lambda) + \mathbf{e}_{\lambda,\pi} \leq T_q - 1 - \alpha + \mathbf{e}_{\lambda,\pi}$$

and conclude using using $\Omega_3^S(\lambda,\pi)$ that i is occupied at time $\mathbf{a}_\lambda T_q$.

Step 3. Finally, point 3 is proved exactly as Point 2. □

We finally examine the $(\lambda,\pi,A)-$FFP around $\lfloor \mathbf{n}_\lambda X_q\rfloor$ at time $\mathbf{a}_\lambda T_q$.

Lemma II.8.7. *We work on $\Omega(\alpha,\gamma,\lambda,\pi) \cap \Omega^{\lambda,\pi}_{T_q}$.*

1. *If $Z_{T_q-}(X_q) < 1$ then there are $j_1, j_2 \in (X_q)_\lambda$ such that $j_1 < \lfloor \mathbf{n}_\lambda X_q\rfloor < j_2$ and $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(j_1) = \eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(j_2) = 0$ for all $s \in [T_q\,, T_q + \kappa^0_{\lambda,\pi}]$.*

2. *If $Z_{T_q-}(X_q) = 1$ then $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_q-}(i) = 1$ for all $i \in [\![\lfloor \mathbf{n}_\lambda (X_q - 2\alpha/p)\rfloor\,, \lfloor \mathbf{n}_\lambda (X_q + 2\alpha/p)\rfloor]\!]$.*

Proof. First observe that $|x - X_q| > 3\alpha/p$ for all $y \in \mathcal{B}^1_{M,q} \cup \mathcal{B}^D_m$ whence $F_{T_q-}(y) = 0$ for all $y \in (X_q - 3\alpha/p\,, X_q + 3\alpha/p)$. We deduce, by Lemma II.8.6, that there is no burning tree in $[\![\lfloor \mathbf{n}_\lambda (X_q - 2\alpha/p)\rfloor\,, \lfloor \mathbf{n}_\lambda (X_q + 2\alpha/p)\rfloor]\!]$ at time $\mathbf{a}_\lambda T_q-$ in the $(\lambda,\pi,A)-$FFP. We distinguish two cases.

Step 1. We first study the case $\tau_{T_q-}(X_q) > 0$. By construction, recalling (II.8.18) and since no match has fallen in X_q during $[0\,, T_q)$, there is a unique $k < q$ such that $\tau_{T_q-}(y) = T_k(y)$ for all $y \in (X_q - 3\alpha/p\,, X_q + 3\alpha/p)$.

If $Z_{T_q-}(X_q) < 1$, then $Z_{T_q-}(X_q) = T_q - \tau_{T_q-}(X_q) < 1$ whence $T_q - \tau_{T_q-}(X_q) < 1 - 3\alpha$, thanks to $\Omega_M(\alpha)$. Recall that for $i \in (X_q)_\lambda$, seeds fall according to $(N^{S,q}_t(i - \lfloor \mathbf{n}_\lambda X_q\rfloor))_{t\geq 0}$.

By Lemma II.8.5, for all $i \in (X_q)_\lambda$,

$$\begin{aligned}\rho^{\lambda,\pi}_{T_q-}(i) \in [T_k + \frac{|i - \lfloor \mathbf{n}_\lambda X_k\rfloor|}{\mathbf{a}_\lambda \pi} - \varepsilon_\lambda\,, T_k + \frac{|i - \lfloor \mathbf{n}_\lambda X_k\rfloor|}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda]\\ \subset (\tau_{T_q-}(X_q) - \mathbf{v}_{\lambda,\pi}\,, \tau_{T_q-}(X_q) + \mathbf{v}_{\lambda,\pi}).\end{aligned}$$

Since we work on $\Omega_2^S(\lambda,\pi)$ and since $T_q, \tau_{T_q-}(X_q) \in \mathcal{B}_M \cup \mathcal{B}^1_{M,q}$, there are some $-\mathbf{m}_\lambda < i_1 < 0 < i_2 < \mathbf{m}_\lambda$ such that no seed has fallen on i_1 and on i_2 during $[\mathbf{a}_\lambda(\tau_{T_q-}(X_q) - 4\mathbf{v}_{\lambda,\pi})\,, \mathbf{a}_\lambda(T_q + 4\mathbf{v}_{\lambda,\pi})] \supset [\mathbf{a}_\lambda T_q\,, \mathbf{a}_\lambda(T_q + \kappa^0_{\lambda,\pi})]$. All this implies that i_1 and i_2 remain vacant during (at least) the time interval $[\mathbf{a}_\lambda T_q\,, \mathbf{a}_\lambda(T_q + \kappa^0_{\lambda,\pi})]$.

If $Z_{T_q-}(X_q) = 1$, then $T_q - \tau_{T_q-}(X_q) \geq 1$ whence $T_q - \tau_{T_q-}(X_q) > 1 + 3\alpha$ and $T_q - \tau_{T_q-}(y) > 1 + \alpha$ for all $y \in (x - 2\alpha/p\,, x + 2\alpha/p)$, thanks to $\Omega_M(\alpha)$.

By Lemma II.8.5, for all $i \in [\![\lfloor \mathbf{n}_\lambda(X_q - 2\alpha/p)\rfloor\,, \lfloor \mathbf{n}_\lambda(X_q + 2\alpha/p)\rfloor]\!]$, we deduce

$$\rho^{\lambda,\pi}_{T_q-}(i) \in [T_k(i/\mathbf{n}_\lambda) - \mathbf{e}_{\lambda,\pi}\,, T_k(i/\mathbf{n}_\lambda) + \mathbf{e}_{\lambda,\pi}].$$

Since we work on $\Omega^S_3(\lambda,\pi)$, at least one seed has fallen on each site during $[\mathbf{a}_\lambda(T_k(i/\mathbf{n}_\lambda) + \mathbf{e}_{\lambda,\pi})\,, \mathbf{a}_\lambda(T_k(i/\mathbf{n}_\lambda) + 1 + \mathbf{e}_{\lambda,\pi})] \subset [\mathbf{a}_\lambda(T_k(i/\mathbf{n}_\lambda) + \mathbf{e}_{\lambda,\pi})\,, \mathbf{a}_\lambda T_q)$. Since, by definition, i cannot been affected by a fire during $(\rho^{\lambda,\pi}_{T_q-}(i)\,, \mathbf{a}_\lambda T_q)$, we deduce that the zone $[\![\lfloor \mathbf{n}_\lambda(X_q - 2\alpha/p)\rfloor\,, \lfloor \mathbf{n}_\lambda(X_q + 2\alpha/p)\rfloor]\!]$ is completely filled at time $\mathbf{a}_\lambda T_q-$.

Step 2. Here we study the case $\tau_{T_q-}(X_q) = 0$. By $\Omega_M(\alpha)$, we have $\tau_{T_q-}(y) = 0$ for all $y \in (X_q - 3\alpha/p\,, X_q + 3\alpha/p)$.

If $Z_{T_q-}(X_q) < 1$, then $Z_{T_q}(X_q) = T_q < 1$ whence $T_q < 1 - 3\alpha$. Since we still work on $\Omega^S_2(\lambda,\pi)$, there are some $-\mathbf{m}_\lambda < i_1 < 0 < i_2 < \mathbf{m}_\lambda$ such that no seed has fallen on i_1 and on i_2 during $[0\,, \mathbf{a}_\lambda(T_q + 4\mathbf{v}_{\lambda,\pi})] \supset [0\,, \mathbf{a}_\lambda(T_q + \kappa^0_{\lambda,\pi})]$. Since we start with a vacant initial configuration, we deduce that i_1 and i_2 remain vacant during (at least) the time interval $[\mathbf{a}_\lambda T_q\,, \mathbf{a}_\lambda(T_q + \kappa^0_{\lambda,\pi})]$.

If $Z_{T_q-}(X_q) = 1$, then $T_q > 1$ whence $T_q > 1 + 3\alpha$. By Lemma II.8.5 we deduce that $\rho^{\lambda,\pi}_{T_q-}(i) = 0$ for all $i \in [\![\lfloor \mathbf{n}_\lambda(X_q - 2\alpha/p)\rfloor\,, \lfloor \mathbf{n}_\lambda(X_q + 2\alpha/p)\rfloor]\!]$ and thus

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_q-}(i) = \min(N^{S,\lambda,\pi}_{\mathbf{a}_\lambda T_q-}(i), 1).$$

Since we work on $\Omega^S_3(\lambda,\pi)$, at least one seed has fallen on each site during $[0\,, \mathbf{a}_\lambda] \subset [0\,, \mathbf{a}_\lambda T_q]$. All this implies that the zone $[\![\lfloor \mathbf{n}_\lambda(X_q - 2\alpha/p)\rfloor\,, \lfloor \mathbf{n}_\lambda(X_q + 2\alpha/p)\rfloor]\!]$ is completely filled at time $\mathbf{a}_\lambda T_q-$. □

The following corollary completes Stage 1.

Corollary II.8.8. *On $\Omega(\alpha,\gamma,\lambda,\pi)$, $\Omega^{\lambda,\pi}_{T_q}$ implies $\Omega^{\lambda,\pi}_{T_q+4\mathbf{v}_{\lambda,\pi}}$.*

Proof. Let $k < q$ such that $T^{D,+}_k \in (T_q\,, T_{q+1})$. By $\Omega_M(\alpha)$, we have $T_q + 3\alpha < T^{D,+}_k$ whence $T_q + 4\mathbf{v}_{\lambda,\pi} < T^{D,+}_k - \mathbf{v}_{\lambda,\pi}$. Thus, no fire extinguishes during $[T_q\,, T_q + 4\mathbf{v}_{\lambda,\pi}]$ (in the limit process). Hence, we have to prove that

- if $X^+_k(T_q) \in \chi^+_{T_q}$, for some $k \leq q$, then $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}(\lfloor \mathbf{n}_\lambda X_k\rfloor + i^{k,+}_{\mathbf{a}_\lambda(t-T_k)}) = 2$ for all $t \in [T_q\,, T_q + 4\mathbf{v}_{\lambda,\pi}]$;
- if $X^-_k(T_q) \in \chi^-_{T_q}$, for some $k \leq q$, then $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}(\lfloor \mathbf{n}_\lambda X_k\rfloor + i^{k,-}_{\mathbf{a}_\lambda(t-T_k)}) = 2$ for all $t \in [T_q\,, T_q + 4\mathbf{v}_{\lambda,\pi}]$;
- if $Z_{T_q-}(X_q) < 1$, then the left and right fronts of the fire ignited at (X_q, T_q) are stopped during the time interval $[\mathbf{a}_\lambda T_q\,, \mathbf{a}_\lambda(T_q + \mathbf{v}_{\lambda,\pi})]$.

Observe that, on $\Omega^{P,T}(\lambda,\pi)$ there a.s. holds that, for all $k \le q$,

$$0 \le i^{k,+}_{\mathbf{a}_\lambda(T_q+4\mathbf{v}_{\lambda,\pi}-T_k)} - i^{k,+}_{\mathbf{a}_\lambda(T_q-T_k)} \le 4(\mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}) \le \lfloor \mathbf{n}_\lambda \alpha/p \rfloor$$

and

$$-\lfloor \mathbf{n}_\lambda \alpha/p \rfloor \le -4(\mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}) \le i^{k,-}_{\mathbf{a}_\lambda(T_q+4\mathbf{v}_{\lambda,\pi}-T_k)} - i^{k,-}_{\mathbf{a}_\lambda(T_q-T_k)} \le 0.$$

All this implies that a front of a fire at time $\mathbf{a}_\lambda T_q$, which belong to $\langle x \rangle_{\lambda,\pi}$ for some $x \in \mathcal{B}^1_{M,q} \cup \{\mathbf{n}_\lambda X_q\}$, can not affect the zone outside $[\![\lfloor \mathbf{n}_\lambda(x-\alpha/p)\rfloor , \lfloor \mathbf{n}_\lambda(x+\alpha/p)\rfloor]\!]$ during the time interval $[\mathbf{a}_\lambda T_q , \mathbf{a}_\lambda(T_q + 4\mathbf{v}_{\lambda,\pi})]$.

Step 1. Here we prove that for $k \le q$ such that $x := X^+_k(T_q) \in \chi^+_{T_q}$ then $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}(\lfloor \mathbf{n}_\lambda X_k \rfloor + i^{k,+}_{\mathbf{a}_\lambda(t-T_k)}) = 2$ for all $t \in [T_q , T_q + 4\mathbf{v}_{\lambda,\pi}]$.

Indeed, by Lemma II.8.6-2 if $k < q$ or by Lemma II.8.7-2 if $k = q$, there holds that

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_q}(\lfloor \mathbf{n}_\lambda X_k \rfloor + i^{k,+}_{\mathbf{a}_\lambda(T_q-T_k)}) = 2$$

and

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_q}(i) = 1 \text{ for all } i \in [\![\lfloor \mathbf{n}_\lambda X_k \rfloor + i^{k,+}_{\mathbf{a}_\lambda(T_q-T_k)} + 1 , \lfloor \mathbf{n}_\lambda(x + 2\alpha/p)\rfloor]\!].$$

But by the previous consideration, no fire, except this one, can affect the zone $[\![\lfloor \mathbf{n}_\lambda X_k \rfloor + i^{k,+}_{\mathbf{a}_\lambda(T_q-T_k)} + 1 , \lfloor \mathbf{n}_\lambda(x+\alpha/p)\rfloor]\!]$ during $[\mathbf{a}_\lambda T_q , \mathbf{a}_\lambda(T_q + 4\mathbf{v}_{\lambda,\pi})]$ and conversely, this fire can not affect the zone outside $[\![\lfloor \mathbf{n}_\lambda(x-\alpha/p)\rfloor , \lfloor \mathbf{n}_\lambda(x+\alpha/p)\rfloor]\!]$. Hence, the right front of the fire k is not stopped during the time interval $[\mathbf{a}_\lambda T_q , \mathbf{a}_\lambda(T_q + 4\mathbf{v}_{\lambda,\pi})]$, as desired.

Step 2. Let $k \le q$, if $x := X^-_k(T_q) \in \chi^-_{T_q}$ then $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}(\lfloor \mathbf{n}_\lambda X_k \rfloor + i^{k,-}_{\mathbf{a}_\lambda(t-T_k)}) = 2$ for all $t \in [T_q , T_q + 4\mathbf{v}_{\lambda,\pi}]$. This can be shown using similar arguments as in Step 1 above.

Step 3. If $Z_{T_q-}(X_q) < 1$, we have $T_q = T^{D,+}_q = T^{D,-}_q$. By Lemma II.8.7-1, we deduce that there are $j_1, j_2 \in (X_q)_\lambda$ such that $j_1 < \lfloor \mathbf{n}_\lambda X_q \rfloor < j_2$ and

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(j_1) = \eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(j_2) = 0 \text{ for all } s \in [T_q , T_q + \kappa^0_{\lambda,\pi}].$$

Hence, on $\Omega^{P,T}_{\lambda,\pi}(X_q, T_q)$, there holds that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_q + T^q_{j_1-\lfloor \mathbf{n}_\lambda X_q \rfloor}}(\lfloor \mathbf{n}_\lambda X_q \rfloor + i^{q,-}_{T^q_{j_1-\lfloor \mathbf{n}_\lambda X_q \rfloor}}) = 0$, because $T_q + T^q_{j_1-\lfloor \mathbf{n}_\lambda X_q \rfloor}/\mathbf{a}_\lambda \le T_q + \kappa^0_{\lambda,\pi}$, and $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_q + T^q_{j_2-\lfloor \mathbf{n}_\lambda X_q \rfloor}}(\lfloor \mathbf{n}_\lambda X_q \rfloor + i^{q,+}_{T^q_{j_1-\lfloor \mathbf{n}_\lambda X_q \rfloor}}) = 0$, because $T_q + T^q_{j_2-\lfloor \mathbf{n}_\lambda X_q \rfloor}/\mathbf{a}_\lambda \le T_q + \kappa^0_{\lambda,\pi}$, as desired. □

STAGE 2.

In this Stage, we assume that $\mathcal{A}_q \neq \emptyset$ and we fix $k \in [\![0 , N_q - 1]\!]$. We work on $\Omega(\alpha, \gamma, \lambda, \pi) \cap \Omega^{\lambda,\pi}_{T^k_q + 4\mathbf{v}_{\lambda,\pi}}$ and prove that $\Omega_{T^{k+1}_q + 4\mathbf{v}_{\lambda,\pi}}$ a.s. holds. We repeatedly use the fact that no match falls in $[-A, A]$ during the time interval $[T_q^k + 4\mathbf{v}_{\lambda,\pi}, T_q^{k+1} + \alpha]$. Observe that, for all $i \in I_A^\lambda$,

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi})-}(i) = \eta^{\lambda,\pi}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi})}(i).$$

We first examine the position of the burning trees of the $(\lambda,\pi,A)-$FFP at time $\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi})$.

Lemma II.8.9. *We work on $\Omega(\alpha,\gamma,\lambda,\pi) \cap \Omega^{\lambda,\pi}_{T_q^k+4\mathbf{v}_{\lambda,\pi}}$.*

1. *At time $\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi})$, a burning tree necessarily belongs to $\langle x\rangle_{\lambda,\pi}$, for some $x \in \chi^+_{T_q^k+4\mathbf{v}_{\lambda,\pi}} \cup \chi^-_{T_q^k+4\mathbf{v}_{\lambda,\pi}}$, and is either $\lfloor \mathbf{n}_\lambda X_l\rfloor + i^{l,+}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi}-T_l)}$ or $\lfloor \mathbf{n}_\lambda X_l\rfloor + i^{l,-}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi}-T_l)}$, for some $l \le q$, or has vacant neighbors.*
2. *If $X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \in \chi^+_{T_q^k+4\mathbf{v}_{\lambda,\pi}}$, for some $l \le q$, then*

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_q^k+4\mathbf{v}_{\lambda,\pi}}(\lfloor \mathbf{n}_\lambda X_l\rfloor + i^{l,+}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi}-T_l)}) = 2$$

   and $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_q^k+4\mathbf{v}_{\lambda,\pi}}(i) = 1$ for all $i \in [\![\lfloor \mathbf{n}_\lambda X_l\rfloor + i^{l,+}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi}-T_l)} + 1, \lfloor \mathbf{n}_\lambda(X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) + 2\alpha/p)\rfloor]\!]$.
3. *If $X_l^-(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \in \chi^-_{T_q^k+4\mathbf{v}_{\lambda,\pi}}$, for some $l \le q$, then $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_q^k+4\mathbf{v}_{\lambda,\pi}}(\lfloor \mathbf{n}_\lambda X_l\rfloor + i^{l,-}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi}-T_l)}) = 2$ and $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_q^k+4\mathbf{v}_{\lambda,\pi}}(i) = 1$ for all $i \in [\![\lfloor \mathbf{n}_\lambda(X_l^-(T_q^k + 4\mathbf{v}_{\lambda,\pi}) - 2\alpha/p)\rfloor, \lfloor \mathbf{n}_\lambda X_l\rfloor + i^{l,-}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi}-T_l)} - 1]\!]$.*

Proof. The proof is very similar to the proof of Lemma II.8.6.

Indeed, we prove point 1 using $\Omega^{P,T}(\lambda,\pi)$ (as in the proof of Lemma II.8.5) which implies that a burning tree necessarily belongs to $\langle X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi})\rangle_{\lambda,\pi}$ or $\langle X_l^-(T_q^k + 4\mathbf{v}_{\lambda,\pi})\rangle_{\lambda,\pi}$ for some $l \le q$ and is either $\lfloor \mathbf{n}_\lambda X_l\rfloor + i^{l,+}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi}-T_l)}$ or $\lfloor \mathbf{n}_\lambda X_l\rfloor + i^{l,-}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi}-T_l)}$ or has vacant neighbors. Furthermore, if $X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) < X_{l'}^-(T_q^k + 4\mathbf{v}_{\lambda,\pi})$, for some $l \neq l'$, we deduce, by $\Omega_M(\alpha)$, that

$$X_{l'}^-(T_q^k + 4\mathbf{v}_{\lambda,\pi}) - X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) > (3\alpha - 8\mathbf{v}_{\lambda,\pi})/p > \frac{5\alpha}{2p}.$$

Thus, as claimed in Step 3 in the proof of Lemma II.8.5, for a site i_0 in $\langle X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi})\rangle_{\lambda,\pi}$ is burning at time $\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi})$, since l is unique, it is necessary that

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_l + T^l_{j-\lfloor \mathbf{n}_\lambda X_l\rfloor}}(j) = 2 \text{ for all } j \in [\![\lfloor \mathbf{n}_\lambda X_l\rfloor, i_0]\!].$$

But, if $X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \notin \chi^+_{T_q^k+4\mathbf{v}_{\lambda,\pi}}$ then $T_l^{D,+} \le T_q^k$. By $\Omega^{\lambda,\pi}_{T_q^k+4\mathbf{v}_{\lambda,\pi}}$, we deduce that there is $j \in [X_l^{D,+}]_{\lambda,\pi}$ such that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_l + T^l_{j-\lfloor \mathbf{n}_\lambda X_l\rfloor}}(j) = 0$ (because there is $s \in$

$[T_l^{D,+} - \mathbf{v}_{\lambda,\pi}\,, T_l^{D,+} + \mathbf{v}_{\lambda,\pi}]$ such that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(\lfloor \mathbf{n}_\lambda X_l \rfloor + i^{l,+}_{\mathbf{a}_\lambda(s-T_l)}) = 0$, recall (II.8.25) and (II.8.26)). Since $\langle X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi})\rangle_{\lambda,\pi} \cap [X_l^{D,+}]_{\lambda,\pi} = \emptyset$, thanks (II.8.29) (recall that $X_l^{D,+} = X_l^+(T_l^{D,+})$), there is no burning tree in $\langle X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi})\rangle_{\lambda,\pi}$ at time $\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi})$.

Point 2 (or point 3) is proved as in Lemma II.8.6. Indeed if $X_l^+(T_q^k+4\mathbf{v}_{\lambda,\pi}) \in \chi^+_{T_q^k+4\mathbf{v}_{\lambda,\pi}}$, then $T_l^{D,+} \geq T_q^{k+1} \geq T_q^k+3\alpha$ and $|X_l^+(T_q^k+4\mathbf{v}_{\lambda,\pi})-y| > 2\alpha$ for all $y \in \mathcal{B}_M^D$. Furthermore, on $\Omega_M(\alpha)$, by construction, we have

$$\tilde{H}_{T_q^k+4\mathbf{v}_{\lambda,\pi}}(y) = 0 \text{ for all } y \in (X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi})\,, X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) + (3\alpha - 4\mathbf{v}_{\lambda,\pi})/p))$$

Thus, we prove that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi})}(j) = 1$ for all $j \in [\![\lfloor \mathbf{n}_\lambda X_l \rfloor + i^{l,+}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi}-T_l)} + 1\,, \lfloor \mathbf{n}_\lambda(X_l^+(T_q^k+4\mathbf{v}_{\lambda,\pi})+2\alpha/p)\rfloor]\!]$ by distinguishing the cases $j \in [\![\lfloor \mathbf{n}_\lambda X_l \rfloor+i^{l,+}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi}-T_l)}+1\,, \lfloor \mathbf{n}_\lambda X_l^+(T_q^k+4\mathbf{v}_{\lambda,\pi})\rfloor+\mathbf{k}_{\lambda,\pi}]\!]$ and $j \in [\![\lfloor \mathbf{n}_\lambda X_l^+(T_q^k+4\mathbf{v}_{\lambda,\pi})\rfloor+\mathbf{k}_{\lambda,\pi}\,, \lfloor \mathbf{n}_\lambda(X_l^+(T_q^k+4\mathbf{v}_{\lambda,\pi})+2\alpha/p)\rfloor]\!]$ (recalling that $X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \notin \mathcal{B}_M^D$). $\square$

We then compute the cluster destroyed by a microscopic fire. We use the notation introduced in Lemma II.8.2.

Lemma II.8.10. *Let $m \leq q$, if $Z_{T_m-}(X_m) < 1$, we define $t_0 = T_m - Z_{T_m-}(X_m)$, which is nothing but $\tau_{T_m-}(X_m)$, recall (II.8.18). We then define, recall (II.8.13) and (II.8.14),*

(i) if $t_0 = T_l(X_m) > 0$ for some $l < m$ and if $X_m = X_l^+(t_0)$,

$$\mathcal{M} \coloneqq (\lfloor \mathbf{n}_\lambda X_l \rfloor + i^{l,+}_{\mathbf{a}_\lambda(t_0-\mathbf{v}_{\lambda,\pi}-T_l)} - \lfloor \mathbf{n}_\lambda X_m \rfloor; t_0, T_m);$$

(ii) if $t_0 = T_l(X_m) > 0$ for some $l < m$ and if $X_m = X_l^-(t_0)$,

$$\mathcal{M} \coloneqq (\lfloor \mathbf{n}_\lambda X_l \rfloor + i^{l,-}_{\mathbf{a}_\lambda(t_0-\mathbf{v}_{\lambda,\pi}-T_l)} - \lfloor \mathbf{n}_\lambda X_m \rfloor; t_0, T_m);$$

(iii) if $t_0 = 0$,

$$\mathcal{M} \coloneqq (0; 0, T_m),$$

Then, working on $\Omega(\alpha, \gamma, \lambda, \pi) \cap \Omega^{\lambda,\pi}_{T_q^k+4\mathbf{v}_{\lambda,\pi}}$, in each case, there holds that

$$(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}(i))_{t\in[t_0-\mathbf{v}_{\lambda,\pi},T_m+\kappa^0_{\lambda,\pi}],i\in(X_m)_\lambda} = (\zeta^{\lambda,\pi,\mathcal{M},m}_{\mathbf{a}_\lambda t}(i - \lfloor \mathbf{n}_\lambda X_m \rfloor))_{t\in[t_0-\mathbf{v}_{\lambda,\pi},T_m+\kappa^0_{\lambda,\pi}],i\in(X_m)_\lambda}$$

where the last process is defined as in Lemma II.8.2 using the seed processes family $(N_t^{S,m}(i))_{t\geq 0,i\in\mathbb{Z}}$ and the propagation processes family $(N_t^{P,m}(i))_{t\geq 0,i\in\mathbb{Z}}$.

This in particular implies that, still on $\Omega(\alpha, \gamma, \lambda, \pi) \cap \Omega^{\lambda,\pi}_{T_q^k+4\mathbf{v}_{\lambda,\pi}}$,

$$C^P((\eta_t^{\lambda,\pi}(i))_{t\geq 0,i\in\mathbb{Z}}, (X_m, T_m)) = [\![\lfloor \mathbf{n}_\lambda X_m \rfloor + i^g\,, \lfloor \mathbf{n}_\lambda X_m \rfloor + i^d]\!] \subset (X_m)_\lambda$$

where $[\![i^g\,, i^d]\!] = C^P((\zeta_t^{\lambda,\pi,\mathcal{M},m}(i))_{t\geq 0,i\in\mathbb{Z}}, (0, T_m)) \subset [\![-\mathbf{m}_\lambda\,, \mathbf{m}_\lambda]\!]$, recall Lemma II.8.2.

Proof. We only treat the case (i). The case (ii) is of course similar and the case (iii) is easier.

We thus fix $1 \leq l < m \leq q$ in such a way that

$$\tau_{T_m-}(X_m) = t_0 = T_l(X_m) \text{ and } X_m = X_l^+(t_0).$$

By $\Omega_M(\alpha)$, we deduce that $T_l^{D,+} > t_0 + 3\alpha$ and $T_m > t_0 + 3\alpha > T_l + 6\alpha$. Hence, by construction, there holds that $Z_{t_0-\mathbf{v}_{\lambda,\pi}}(y) = 1$ for all $y \in (X_m - \mathbf{v}_{\lambda,\pi}/p \,, X_m + 2\alpha/p)$. Observe that $T_q^k + 4\mathbf{v}_{\lambda,\pi} \geq T_m + \kappa^0_{\lambda,\pi}$.

By $\Omega^{\lambda,\pi}_{T_q^k+4\mathbf{v}_{\lambda,\pi}}$, we deduce that at time $\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi})$ the site

$$\lfloor \mathbf{n}_\lambda X_l \rfloor + i^{l,+}_{\mathbf{a}_\lambda(t_0-\mathbf{v}_{\lambda,\pi}-T_l)} \in [\![\lfloor \mathbf{n}_\lambda X_m \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi} \,, \lfloor \mathbf{n}_\lambda X_m \rfloor - \mathbf{m}_\lambda]\!]$$

is burning whereas the zone $[\![\lfloor \mathbf{n}_\lambda X_l \rfloor + i^{l,+}_{\mathbf{a}_\lambda(t_0-\mathbf{v}_{\lambda,\pi}-T_l)} + 1 \,, \lfloor \mathbf{n}_\lambda (X_m + 2\alpha/p) \rfloor]\!]$ is completely occupied (use very similar arguments as in Lemma II.8.9-2, recalling that no match falls on X_m during $[0\,, T_m) \supset [0\,, t_0)$). Comparing $(\eta_t^{\lambda,\pi}(i))_{t\geq 0, i\in\mathbb{Z}}$ and $(\zeta_t^{\lambda,\pi,\mathcal{M},m}(i - \lfloor \mathbf{n}_\lambda X_m \rfloor))_{t\geq 0, i\in\mathbb{Z}}$, we deduce that they are equal on $[\![\lfloor \mathbf{n}_\lambda X_l \rfloor + i^{k,+}_{\mathbf{a}_\lambda(t_0-\mathbf{v}_{\lambda,\pi}-T_l)} \,, \lfloor \mathbf{n}_\lambda X_m \rfloor + \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}]\!] \supset (X_m)_\lambda$ at time $\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi})$.

Since, with our coupling, seeds fall according to the same processes and fires spread according to the same processes on $[X_m]_{\lambda,\pi}$, we deduce that the fire preads in the same way through $[\![\lfloor \mathbf{n}_\lambda X_l \rfloor + i^{k,+}_{\mathbf{a}_\lambda(t_0-\mathbf{v}_{\lambda,\pi}-T_l)} \,, \lfloor \mathbf{n}_\lambda X_m \rfloor + \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}]\!]$. Thus, $(\eta_t^{\lambda,\pi}(i))_{t\geq 0, i\in\mathbb{Z}}$ and $(\zeta_t^{\lambda,\pi,\mathcal{M},m}(i - \lfloor \mathbf{n}_\lambda X_m \rfloor))_{t\geq 0, i\in\mathbb{Z}}$ remain equal on $[\![\lfloor \mathbf{n}_\lambda X_l \rfloor + i^{k,+}_{\mathbf{a}_\lambda(t_0-\mathbf{v}_{\lambda,\pi}-T_l)} + 1 \,, \lfloor \mathbf{n}_\lambda X_m \rfloor + \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}]\!] \supset (X_m)_\lambda$ during the time interval $[\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi}) \,, \mathbf{a}_\lambda(t_0 + 4\mathbf{v}_{\lambda,\pi})]$, recall (II.8.29). No other fire affect the zone $(X_m)_\lambda$ until a match falls on $\lfloor \mathbf{n}_\lambda X_m \rfloor$ at time $\mathbf{a}_\lambda T_m$ because the zone $(X_m)_\lambda$ is protected by vacant site during the time interval $[\mathbf{a}_\lambda(t_0 + 4\mathbf{v}_{\lambda,\pi}) \,, \mathbf{a}_\lambda(T_m + \kappa^0_{\lambda,\pi})]$ (by construction for $\zeta^{\lambda,\pi,\mathcal{M},m}$ and because in the $(\lambda, \pi, A)-$FFP, on $\Omega_2^S(\lambda,\pi)$, there are

$$-\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi} < i_1 < -\mathbf{m}_\lambda < \mathbf{m}_\lambda < i_2 < \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}$$

where no seed fall during the time interval $(\mathbf{a}_\lambda(t_0 - 4\mathbf{v}_{\lambda,\pi}) \,, \mathbf{a}_\lambda(T_m + \kappa^0_{\lambda,\pi}))$ and because the sites $\lfloor \mathbf{n}_\lambda X_m \rfloor + i_1$ and $\lfloor \mathbf{n}_\lambda X_m \rfloor + i_2$ has been made vacant by the fire l during $(\mathbf{a}_\lambda(t_0 - 4\mathbf{v}_{\lambda,\pi}) \,, \mathbf{a}_\lambda(t_0 + 4\mathbf{v}_{\lambda,\pi}))$, recall (II.8.28) and (II.8.29)). Thus, since seeds fall on $[X_m]_{\lambda,\pi}$ according to the same processes, $(\eta_t^{\lambda,\pi}(i))_{t\geq 0, i\in\mathbb{Z}}$ and $(\zeta_t^{\lambda,\pi,\mathcal{M},m}(i - \lfloor \mathbf{n}_\lambda X_m \rfloor))_{t\geq 0, i\in\mathbb{Z}}$ remain equal on $(X_m)_\lambda$ during $[\mathbf{a}_\lambda(t_0 + 4\mathbf{v}_{\lambda,\pi}) \,, \mathbf{a}_\lambda T_m)$. Finally, by $\Omega_2^S(\lambda,\pi)$, we deduce that there are some sites

$$-\mathbf{m}_\lambda < i_3 < 0 < i_4 < \mathbf{m}_\lambda$$

where no seed fall during the time interval $[\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi}) \,, \mathbf{a}_\lambda(T_m + \kappa^0_{\lambda,\pi})]$ whence, as usual, in both cases, the sites $\lfloor \mathbf{n}_\lambda X_m \rfloor + i_3$ and $\lfloor \mathbf{n}_\lambda X_m \rfloor + i_4$ are vacant during $[\mathbf{a}_\lambda(t_0 + \mathbf{v}_{\lambda,\pi}) \,, \mathbf{a}_\lambda(T_m + \kappa^0_{\lambda,\pi})]$, recall (II.8.26) (because they are made vacant by the fire l). Since the two processes evolve according to the same rules, the match falling on $\lfloor \mathbf{n}_\lambda X_m \rfloor$ at time

$\mathbf{a}_\lambda T_m$ destroys the same zone. Thus, $(\eta_t^{\lambda,\pi}(i))_{t\geq 0,i\in\mathbb{Z}}$ and $(\zeta_t^{\lambda,\pi,\mathcal{M},m}(i-\lfloor \mathbf{n}_\lambda X_m\rfloor))_{t\geq 0,i\in\mathbb{Z}}$ are also equal on $(X_m)_\lambda$ during $[\mathbf{a}_\lambda T_m\,,\mathbf{a}_\lambda(T_m+\kappa^0_{\lambda,\pi})]$.

We deduce, on $\Omega_2^S(\lambda,\pi)$, as seen in **Micro**(p) in Subsection II.4.4, that

$$C^P((\zeta_t^{\lambda,\pi,\mathcal{M},m}(i))_{t\geq 0,i\in\mathbb{Z}},(0,T_m)) \coloneqq [\![i^g\,,i^d]\!] \subset [\![-\mathbf{m}_\lambda\,,\mathbf{m}_\lambda]\!]$$

and that there is no more burning tree in $(X_m)_\lambda$ at time $\mathbf{a}_\lambda(T_m+\kappa^0_{\lambda,\pi})$, whence

$$C^P((\eta_t^{\lambda,\pi}(i))_{t\geq 0,i\in\mathbb{Z}},(X_m,T_m)) = [\![\lfloor \mathbf{n}_\lambda X_m\rfloor + i^g\,,\lfloor \mathbf{n}_\lambda X_m\rfloor + i^d]\!] \subset (X_m)_\lambda. \qquad \square$$

We will need the following lemma.

Lemma II.8.11. *Let $s_0 \in [T_q^k+\alpha\,,T_q^{k+1}+\alpha]$. We work on $\Omega(\alpha,\gamma,\lambda,\pi)\cap\Omega^{\lambda,\pi}_{T_q^k+4\mathbf{v}_{\lambda,\pi}}$.*

1. *In the limit process, if, for some $l\leq q$, $X_l^+(T_q^k+4\mathbf{v}_{\lambda,\pi})\in\chi^+_{T_q^k+4\mathbf{v}_{\lambda,\pi}}$ in such a way that $s_0\leq T_l^{D,+}$ and*

$$F_{T_q^k+4\mathbf{v}_{\lambda,\pi}}(y)=0 \text{ for all } y\in(X_l^+(T_q^k+4\mathbf{v}_{\lambda,\pi})\,,X_l^+(s_0+\alpha)), \tag{II.8.33}$$

then, in the discrete process, the site $\lfloor \mathbf{n}_\lambda X_l^+(s_0)\rfloor$ is not affected by a fire during the time interval $[\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi})\,,\mathbf{a}_\lambda(s_0-\mathbf{e}_{\lambda,\pi})]$.

2. *In the limit process, if, for some $l\leq q$, $X_l^-(T_q^k+4\mathbf{v}_{\lambda,\pi})\in\chi^-_{T_q^k+4\mathbf{v}_{\lambda,\pi}}$ in such a way that $s_0\leq T_l^{D,-}$ and $F_{T_q^k+4\mathbf{v}_{\lambda,\pi}}(y)=0$ for all $y\in(X_l^-(s_0+\alpha)\,,X_l^-(T_q^k+4\mathbf{v}_{\lambda,\pi}))$, then, in the discrete process, the site $\lfloor \mathbf{n}_\lambda X_l^-(s_0)\rfloor$ is not affected by a fire during the time interval $[\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi})\,,\mathbf{a}_\lambda(s_0-\mathbf{e}_{\lambda,\pi})]$.*

Proof. It of course suffices to prove 1.

First, using (II.8.33), we deduce that

$$(X_l^+(T_q^k+4\mathbf{v}_{\lambda,\pi})\,,X_l^+(s_0+\alpha))\cap\left(\chi^+_{T_q^k+4\mathbf{v}_{\lambda,\pi}}\cup\chi^-_{T_q^k+4\mathbf{v}_{\lambda,\pi}}\right)=\emptyset.$$

Hence, by Lemma II.8.9-1 and by (II.8.24), we deduce that there is no burning tree in $[\![\lfloor \mathbf{n}_\lambda X_l\rfloor + i^{l,+}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi}-T_l)}+1\,,\lfloor \mathbf{n}_\lambda X_l^+(s_0+\alpha)\rfloor-\mathbf{k}_{\lambda,\pi}]\!]$ at time $\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi})$.

On the one hand, on $\Omega(\alpha,\gamma,\lambda,\pi)$, recall (II.8.27) and Lemma II.4.2, there holds that

$$\lfloor \mathbf{n}_\lambda X_l\rfloor + i^{l,+}_{\mathbf{a}_\lambda(s_0-\mathbf{e}_{\lambda,\pi}-T_l)} < \lfloor \mathbf{n}_\lambda X_l^+(s_0)\rfloor.$$

Thus the right front of the fire l does not reach $\lfloor \mathbf{n}_\lambda X_l^+(s_0)\rfloor$ before $\mathbf{a}_\lambda(s_0-\mathbf{e}_{\lambda,\pi})$. Hence, no fire coming from the left can affect the site $\lfloor \mathbf{n}_\lambda X_l^+(s_0)\rfloor$ during the considered time interval.

On the other hand, no fire coming from the right can affect $\lfloor \mathbf{n}_\lambda X_l^+(s_0)\rfloor$ before $\mathbf{a}_\lambda(s_0-\mathbf{e}_{\lambda,\pi})$. Indeed, since there is no fire in $[\![\lfloor \mathbf{n}_\lambda X_l^+(s_0)\rfloor\,,\lfloor \mathbf{n}_\lambda X_l^+(s_0+\alpha)\rfloor-\mathbf{k}_{\lambda,\pi}]\!]$ at time $\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi})$, we deduce, by $\Omega(\alpha,\gamma,\lambda,\pi)$, that if a fire affect the site $\lfloor \mathbf{n}_\lambda X_l^+(s_0)\rfloor$

during the time interval $[\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi}) , \mathbf{a}_\lambda(s_0 - \mathbf{e}_{\lambda,\pi})]$, it is necessarily a left front. But, by construction, if $X_{l'}^-(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \in \chi^-_{T_q^k+4\mathbf{v}_{\lambda,\pi}}$, for some $l' \leq q$, then $X_l^+(s_0) \leq X_{l'}^-(s_0)$ (because $s_0 \leq T_l^{D,+}$). By (II.8.27) and Lemma II.4.2, we then have

$$\lfloor \mathbf{n}_\lambda X_{l'} \rfloor + i^{l',-}_{\mathbf{a}_\lambda(s_0 - \mathbf{e}_{\lambda,\pi} - T_{l'})} > \lfloor \mathbf{n}_\lambda X_{l'}^-(s_0) \rfloor \geq \lfloor \mathbf{n}_\lambda X_l^-(s_0) \rfloor.$$

Hence, no fire coming from the right can affect $\lfloor \mathbf{n}_\lambda X_l^+(s_0) \rfloor$ during the considered time interval. ☐

The two following lemmas are the keys of this Stage. The first of them insure that a fire indeed propagates. The second insure that a fire is stopped when it meet a microscopic zone.

Lemma II.8.12. *Let $s_0 \in [T_q^k + \alpha , T_q^{k+1} + \alpha]$. We work on $\Omega(\alpha, \gamma, \lambda, \pi) \cap \Omega^{\lambda,\pi}_{T_q^k+4\mathbf{v}_{\lambda,\pi}}$.*

1. *In the limit process, if $X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \in \chi^+_{T_q^k+4\mathbf{v}_{\lambda,\pi}}$ for some $l \leq q$ in such a way that $s_0 \leq T_l^{D,+}$ and $F_{T_q^k+4\mathbf{v}_{\lambda,\pi}}(y) = 0$ for all $y \in (X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) , X_l^+(s_0 + \alpha))$, then*
$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_l + T^l_{i - \lfloor \mathbf{n}_\lambda X_l \rfloor}}(i) = 2$$
for all $i \in [\![\lfloor \mathbf{n}_\lambda X_l \rfloor + i^{l,+}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi} - T_l)} , \lfloor \mathbf{n}_\lambda X_l^+(s_0) \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}]\!]$.

2. *In the limit process, if $X_l^-(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \in \chi^-_{T_q^k+4\mathbf{v}_{\lambda,\pi}}$ for all $y \in (X_l^-(s_0+\alpha) , X_l^-(T_q^k + 4\mathbf{v}_{\lambda,\pi}))$, then $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_l + T^l_{i - \lfloor \mathbf{n}_\lambda X_l \rfloor}}(i) = 2$ for all $i \in [\![\lfloor \mathbf{n}_\lambda X_l^-(s_0) \rfloor + \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi} , \lfloor \mathbf{n}_\lambda X_l \rfloor + i^{l,-}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi} - T_l)}]\!]$.*

We can prove the propagation of the fire l only to $\lfloor \mathbf{n}_\lambda X_l^+(s_0) \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}$. Unfortunately, if $s_0 = T_q^{k+1} = T_l^{D,+}$ and $X_l^+(T_q^{k+1}) = X_q^{k+1} = X_l^{D,+}$ (that is if the right front of the fire l is stopped at time T_q^{k+1} in the limit process), we can not say anything more on the discrete process without a careful study of the process. We will show below (see Lemma II.8.13) that, in this special case, the zone $[\![\lfloor \mathbf{n}_\lambda X_q^{k+1} \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi} , \lfloor \mathbf{n}_\lambda X_q^{k+1} \rfloor - \mathbf{m}_\lambda]\!]$ is actually completely occupied at time $\mathbf{a}_\lambda(T_q^{k+1} - 4\mathbf{v}_{\lambda,\pi})$. This will imply that the fire propagates indeed until $\mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda,\pi})$, thanks to (II.8.25).

Proof. Lemma II.8.11 shows that the site $\lfloor \mathbf{n}_\lambda X_l^+(s_0) \rfloor$ is not affected by a fire during $[\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi}) , \mathbf{a}_\lambda(s_0 - \mathbf{e}_{\lambda,\pi})]$. Hence, no fire coming from the right affect the zone $[\![\lfloor \mathbf{n}_\lambda X_l \rfloor + i^{l,+}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi} - T_l)} + 1 , \lfloor \mathbf{n}_\lambda X_l^+(s_0) \rfloor]\!]$ during the time interval $[\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi}) , \mathbf{a}_\lambda(s_0 - \mathbf{v}_{\lambda,\pi})]$ and, conversely, the right front of the fire l does not affect the zone on the right of $\lfloor \mathbf{n}_\lambda X_l^+(s_0) \rfloor$. Since $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi})}(\lfloor \mathbf{n}_\lambda X_l \rfloor + i^{l,+}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi} - T_l)}) = 2$, thanks to Lemma II.8.9-2, it then suffices to show that for all $i \in [\![\lfloor \mathbf{n}_\lambda X_l \rfloor + i^{l,+}_{\mathbf{a}_\lambda(T^k_q+4\mathbf{v}_{\lambda,\pi}-T_l)} + 1\,,\lfloor \mathbf{n}_\lambda X^+_l(s_0)\rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}]\!]$,

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_l + T^l_{i-\lfloor \mathbf{n}_\lambda X_l\rfloor}-}(i) = 1$$

i.e. the site i is occupied just before that the right front of the fire l reaches i.

Observe that by construction, in the limit process, no fire affect the site $i/\mathbf{n}_\lambda \in (X^+_l(T^k_q + 4\mathbf{v}_{\lambda,\pi})\,, X^+_l(s_0))$ during $(T^k_q + 4\mathbf{v}_{\lambda,\pi}\,, T_l(i/\mathbf{n}_\lambda))$ whence in the discrete process, no fire can affect the site $i \in [\![\lfloor \mathbf{n}_\lambda X_l \rfloor + i^{l,+}_{\mathbf{a}_\lambda(T^k_q+4\mathbf{v}_{\lambda,\pi}-T_l)} + 1\,,\lfloor \mathbf{n}_\lambda X^+_l(s_0)\rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}]\!]$ during $[\mathbf{a}_\lambda(T^k_q + 4\mathbf{v}_{\lambda,\pi})\,, \mathbf{a}_\lambda T_l + T^l_{i-\lfloor \mathbf{n}_\lambda X_l\rfloor})$. All this implies that for all $i/\mathbf{n}_\lambda \in (X^+_l(T^k_q + 4\mathbf{v}_{\lambda,\pi})\,, X^+_l(s_0))$, we have

$$\tau_{T_l(i/\mathbf{n}_\lambda)-}(i/\mathbf{n}_\lambda) = \tau_{T^k_q+4\mathbf{v}_{\lambda,\pi}}(i/\mathbf{n}_\lambda)$$

while for all $i \in [\![\lfloor \mathbf{n}_\lambda X_l \rfloor + i^{l,+}_{\mathbf{a}_\lambda(T^k_q+4\mathbf{v}_{\lambda,\pi}-T_l)} + 1\,,\lfloor \mathbf{n}_\lambda X^+_l(s_0)\rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}]\!]$ we have

$$\rho^{\lambda,\pi}_{T_l + T^l_{i-\lfloor \mathbf{n}_\lambda X_l\rfloor}/\mathbf{a}_\lambda -}(i) = \rho^{\lambda,\pi}_{T^k_q+4\mathbf{v}_{\lambda,\pi}}(i).$$

Step 1. Here we show that for all $j \in [\![\lfloor \mathbf{n}_\lambda X_l \rfloor + i^{l,+}_{\mathbf{a}_\lambda(T^k_q+4\mathbf{v}_{\lambda,\pi}-T_l)} + 1\,,\lfloor \mathbf{n}_\lambda X^+_l(T^k_q + 4\mathbf{v}_{\lambda,\pi})\rfloor + \mathbf{k}_{\lambda,\pi}]\!]$, we have $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_l + T^l_{j-\lfloor \mathbf{n}_\lambda X_l\rfloor}-}(j) = 1$.

In Lemma II.8.9-2 we have proved that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda(T^k_q+4\mathbf{v}_{\lambda,\pi})}(j) = 1$ for all $j \in [\![\lfloor \mathbf{n}_\lambda X_l \rfloor + i^{l,+}_{\mathbf{a}_\lambda(T^k_q+4\mathbf{v}_{\lambda,\pi}-T_l)} + 1\,,\lfloor \mathbf{n}_\lambda X^+_l(T^k_q + 4\mathbf{v}_{\lambda,\pi})\rfloor + \mathbf{k}_{\lambda,\pi}]\!]$. The result follows from the previous observation.

Step 2. Here we show that for all $j \in [\![\lfloor \mathbf{n}_\lambda X^+_l(T^k_q + 4\mathbf{v}_{\lambda,\pi})\rfloor + \mathbf{k}_{\lambda,\pi} + 1\,,\lfloor \mathbf{n}_\lambda X^+_l(s_0)\rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}]\!] \setminus \cup_{y\in\mathcal{B}^D_M}[y]_{\lambda,\pi}$, we have $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_l + T^l_{j-\lfloor \mathbf{n}_\lambda X_l\rfloor}-}(j) = 1$.

Indeed, on the one hand, $Z_{T_l(j/\mathbf{n}_\lambda)-}(j/\mathbf{n}_\lambda) = 1$, then $T_l(j/\mathbf{n}_\lambda) - \tau_{T_l(j/\mathbf{n}_\lambda)-}(j/\mathbf{n}_\lambda) > 1$ whence

$$\tau_{T_l(j/\mathbf{n}_\lambda)-}(j/\mathbf{n}_\lambda) < T_l(j/\mathbf{n}_\lambda) - 1 - 3\alpha,$$

thanks to $\Omega_M(\alpha)$. On the other hand, recalling that there is no burning tree in $[\![\lfloor \mathbf{n}_\lambda X^+_l(T^k_q + 4\mathbf{v}_{\lambda,\pi})\rfloor + \mathbf{k}_{\lambda,\pi} + 1\,,\lfloor \mathbf{n}_\lambda X^+_l(s_0)\rfloor]\!]$ at time $\mathbf{a}_\lambda(T^k_q + 4\mathbf{v}_{\lambda,\pi})$ (and thus $j \notin \bigcup_{x\in\mathcal{X}_{T^k_q+4\mathbf{v}_{\lambda,\pi}}}\langle x\rangle_{\lambda,\pi}$) and since $j \notin \bigcup_{x\in\mathcal{B}^D_M}[x]_{\lambda,\pi}$, we deduce from Lemma II.8.5 and by (II.8.30) that

$$\rho^{\lambda,\pi}_{T^k_q+4\mathbf{v}_{\lambda,\pi}}(j) \le \tau_{T^k_q+4\mathbf{v}_{\lambda,\pi}}(j/\mathbf{n}_\lambda) + \mathbf{e}_{\lambda,\pi}.$$

All this implies that

$$\rho^{\lambda,\pi}_{T_l + T^l_{j-\lfloor \mathbf{n}_\lambda X_l\rfloor}/\mathbf{a}_\lambda -}(j) \le T_l(j/\mathbf{n}_\lambda) - 1 - 3\alpha + \mathbf{e}_{\lambda,\pi}.$$

Recalling that $T_l + T^l_{j-\lfloor \mathbf{n}_\lambda X_l\rfloor}/\mathbf{a}_\lambda \ge T_l(j/\mathbf{n}_\lambda) - \mathbf{e}_{\lambda,\pi}$, thanks to (II.8.30), and $\mathbf{e}_{\lambda,\pi} < \alpha$, we conclude using $\Omega^S_3(\lambda,\pi)$ that the site j is occupied at time $\mathbf{a}_\lambda T_l + T^l_{j-\lfloor \mathbf{n}_\lambda X_l\rfloor}-$.

Step 3. Here we show that for all $y \in \mathcal{B}_M^D \cap (X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}), X_l^+(s_0))$, for all $j \in [y]_{\lambda,\pi}$, there holds $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda(T_l(y)-4\mathbf{v}_{\lambda,\pi})}(j) = 1$. This will conclude Lemma II.8.12 since $\mathbf{a}_\lambda T_l + T^l_{j-\lfloor \mathbf{n}_\lambda X_l \rfloor} \geq \mathbf{a}_\lambda(T_l(y) - 4\mathbf{v}_{\lambda,\pi})$ for all $j \in [y]_{\lambda,\pi}$, thanks to (II.8.28).

Preliminary considerations. Let $y \in \mathcal{B}_M^D \cap (X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}), X_l^+(s_0))$. Since $X_l^+(s_0) \leq X_l^{D,+}$, we have $y \leq X_l^{D,+} - 3\alpha/p$. We may assume $X_l^+(s_0) \geq y + \alpha/p$, by $\Omega_M(\alpha)$. We know that $\tilde{H}_{T_l(y)-}(y) = 0$, whence $H_{T_l(y)-}(y) = 0$ and $Z_{T_l(y)-}(y) = Z_{T_l(y)-}(y_+) = Z_{T_l(y)-}(y_-) = 1$. This implies that $T_l(y) \geq 1$ (because $Z_t(y) = t$ for all $t < 1$ and all $y \in [-A, A]$).

As pointed out in Step 2, we have, setting $j_g = \lfloor \mathbf{n}_\lambda y \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi} - 1$ and observing that $T_l + T^l_{j_g - \lfloor \mathbf{n}_\lambda X_l \rfloor}/\mathbf{a}_\lambda \geq T_l(y) - 4\mathbf{v}_{\lambda,\pi} \geq T_q^k + 4\mathbf{v}_{\lambda,\pi}$,

$$\rho^{\lambda,\pi}_{T_l(y)-4\mathbf{v}_{\lambda,\pi}}(j_g) \leq T_l(j_g/\mathbf{n}_\lambda) - 1 - 3\alpha + \mathbf{e}_{\lambda,\pi} = T_l(y) - 1 - 3\alpha + \mathbf{e}_{\lambda,\pi} - p\frac{\mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi} + 1}{\mathbf{n}_\lambda}.$$

Using a similar argument for $j_d = \lfloor \mathbf{n}_\lambda y \rfloor + \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi} + 1$, we conclude that no match falling outside $[y]_{\lambda,\pi} = [\![j_g + 1, j_d - 1]\!]$ can affect $[y]_{\lambda,\pi}$ during $(\mathbf{a}_\lambda(T_l(y) - 1 - \alpha), \mathbf{a}_\lambda(T_l(y) - 4\mathbf{v}_{\lambda,\pi}))$, because

$$\rho^{\lambda,\pi}_{T_l(y)-4\mathbf{v}_{\lambda,\pi}}(j_g) + 2\varepsilon_\lambda + 2\frac{\mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}}{\mathbf{a}_\lambda \pi} \leq T_l(y) - 1 - \alpha$$

and because to affect a site $i \in [y]_{\lambda,\pi}$, a match falling outside $[y]_{\lambda,\pi}$ needs to cross j_d or j_g and thus must verify, recall Lemma II.8.5,

$$\rho^{\lambda,\pi}_{T_l(y)-4\mathbf{v}_{\lambda,\pi}}(i) \leq (\rho^{\lambda,\pi}_{T_l(y)-4\mathbf{v}_{\lambda,\pi}}(j_g/\mathbf{n}_\lambda) \vee \rho^{\lambda,\pi}_{T_l(y)-4\mathbf{v}_{\lambda,\pi}}(j_d/\mathbf{n}_\lambda)) + 2(\kappa^0_{\lambda,\pi} + \mathbf{e}_{\lambda,\pi}).$$

Case 1. First assume that $y \in \mathcal{B}_M^2$. Then we know that no match has fallen on $[y]_{\lambda,\pi}$ during $[0, \mathbf{a}_\lambda T_l(y))$. Due to the preliminary considerations, we deduce that no fire at all has concerned $[y]_{\lambda,\pi}$ during $(\mathbf{a}_\lambda(T_l(y) - 1 - \alpha), \mathbf{a}_\lambda(T_l(y) - 4\mathbf{v}_{\lambda,\pi}))$. Using $\Omega_3^S(\lambda, \pi)$, we conclude that $[y]_{\lambda,\pi}$ is completely occupied at time $\mathbf{a}_\lambda(T_l(y) - 4\mathbf{v}_{\lambda,\pi})$.

Case 2. Assume that $y = X_m \in \mathcal{B}_M$ with $m \geq q + 1$. Then we know that no match has fallen on $[X_m]_{\lambda,\pi}$ during $[0, \mathbf{a}_\lambda T_l(X_m)) \subset [0, \mathbf{a}_\lambda T_m)$. We conclude as in Case 1 using $\Omega_3^S(\lambda, \pi)$ that the zone $[X_m]_{\lambda,\pi}$ is completely occupied at time $\mathbf{a}_\lambda(T_l(y) - 4\mathbf{v}_{\lambda,\pi})$.

Case 3. Assume that $y = X_m \in \mathcal{B}_M$ with $m \leq q$ and $Z_{T_m-}(X_m) = 1$, so that there already has been a macroscopic fire in $[X_m]_{\lambda,\pi}$ (at time $\mathbf{a}_\lambda T_m$). There is no more burning tree in $[X_m]_{\lambda,\pi}$ at time $\mathbf{a}_\lambda(T_m + 4\mathbf{v}_{\lambda,\pi})$, thanks to $\Omega^{P,T}_{\lambda,\pi}(X_m, T_m)$ and (II.8.29). Since $Z_{T_m}(X_m) = 0$ and $Z_{T_l(X_m)-}(X_m) = 1$, we deduce that $T_l(X_m) - T_m \geq 1$, whence $T_l(X_m) - T_m \geq 1 + 3\alpha$ as usual. We conclude as in case 1 that no fire at all has concerned $[X_m]_{\lambda,\pi}$ during $(\mathbf{a}_\lambda(T_l(X_m) - 1 - \alpha), \mathbf{a}_\lambda(T_l(X_m) - 4\mathbf{v}_{\lambda,\pi}))$, which implies the claim by $\Omega_3^S(\lambda, \pi)$.

Case 4. Assume that $y = X_m \in \mathcal{B}_M$ with $m \leq q$ and $Z_{T_m-}(X_m) < 1$ and $T_l(X_m) - T_m \geq 1$, whence $T_l(X_m) - T_m \geq 1 + 3\alpha$ due to $\Omega_M(\alpha)$. Then there already has been a microscopic fire in $[X_m]_{\lambda,\pi}$ (at time $\mathbf{a}_\lambda T_m$). There is no more burning tree in $[X_m]_{\lambda,\pi}$ at time $\mathbf{a}_\lambda(T_m+4\mathbf{v}_{\lambda,\pi})$, thanks to $\Omega^{P,T}_{\lambda,\pi}(X_m,T_m)$ and (II.8.29). No match falls on $[X_m]_{\lambda,\pi}$ during $(\mathbf{a}_\lambda(T_m+4\mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(T_l(X_m)-4\mathbf{v}_{\lambda,\pi})) \supset (\mathbf{a}_\lambda(T_l(X_m)-1-\alpha), \mathbf{a}_\lambda(T_l(X_m)-4\mathbf{v}_{\lambda,\pi}))$ and we conclude as in case 1.

Case 5. Assume that $y = X_m \in \mathcal{B}_M$ with $m \le q$ and $Z_{T_m-}(X_m) < 1$ and $T_l(X_m) - T_m < 1$, whence $T_l(X_m)-T_m \le 1-3\alpha$ due to $\Omega_M(\alpha)$. There has been a microscopic fire in $[X_m]_{\lambda,\pi}$ (at time $\mathbf{a}_\lambda T_m$). Since $H_{T_l(X_m)}(X_m) = 0$, we deduce that $T_m + Z_{T_m-}(X_m) \le T_l(X_m)$, whence $T_m + Z_{T_m-}(X_m) \le T_l(X_m) - 3\alpha$ by $\Omega_M(\alpha)$. We define $\mathcal{M} = (i_0; t_0, T_m)$ as in Lemma II.8.10.

Consider the zone $C^P := C^P((\eta^{\lambda,\pi}_t(i))_{t\ge 0, i\in\mathbb{Z}}, (X_m,T_m)) \subset (X_m)_\lambda$ destroyed by the match falling on $\lfloor \mathbf{n}_\lambda X_m \rfloor$ at time $\mathbf{a}_\lambda T_m$. This zone is completely occupied at time $\mathbf{a}_\lambda(T_m + \Theta^{\lambda,\pi,m}_{\mathcal{M}})$: this follows from the definition of $\Theta^{\lambda,\pi,m}_{\mathcal{M}}$ (see Lemma II.8.2), from Lemma II.8.10 and from the preliminary considerations (because $T_m \ge T_l(X_m) - 1 - \alpha$). Using $\Omega^S_4(\gamma,\lambda,\pi)$, we deduce that $T_m + \Theta^{\lambda,\pi,m}_{\mathcal{M}} \le T_m + Z_{T_m-}(X_m) + \gamma < T_l(X_m) - 2\alpha$, since $\gamma < \alpha$. Hence C^P is completely occupied at time $\mathbf{a}_\lambda(T_l(X_m) - 4\mathbf{v}_{\lambda,\pi})$.

Consider now $i \in [X_m]_{\lambda,\pi} \setminus C^P$. Then i has not been killed by the fire starting at $\lfloor \mathbf{n}_\lambda X_m \rfloor$. Thus i cannot have been killed during $(\mathbf{a}_\lambda(T_l(X_m)-1-\alpha), \mathbf{a}_\lambda(T_l(X_m) - 4\mathbf{v}_{\lambda,\pi}))$ (due to the preliminary considerations) and we conclude, using $\Omega^S_3(\lambda,\pi)$, that i is occupied at time $\mathbf{a}_\lambda(T_l(X_m) - 4\mathbf{v}_{\lambda,\pi})$. This implies the claim. □

We now examine the process at time $\mathbf{a}_\lambda T^{k+1}_q$ around $\lfloor \mathbf{n}_\lambda X^{k+1}_q \rfloor$ in the case where the fire is stopped by a microscopic zone (in the limit process).

Lemma II.8.13. *On $\Omega(\alpha,\gamma,\lambda,\pi) \cap \Omega^{\lambda,\pi}_{T^k_q+4\mathbf{v}_{\lambda,\pi}}$, if $F_{T^{k+1}_q}(X^{k+1}_q) \le 1$, there exists $i \in (X^{k+1}_q)_\lambda$ such that*

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(i) = 0 \text{ for all } s \in [T^{k+1}_q - \mathbf{v}_{\lambda,\pi}, T^{k+1}_q + \mathbf{v}_{\lambda,\pi}].$$

Furthermore,

(i) if $X^{k+1}_q = X^+_l(T^{k+1}_q)$ for some $l \le q$, then the zone

$$[\![\lfloor \mathbf{n}_\lambda X^{k+1}_q \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}, \lfloor \mathbf{n}_\lambda X^{k+1}_q \rfloor - \mathbf{m}_\lambda]\!]$$

is completely occupied at time $\mathbf{a}_\lambda(T^{k+1}_q - 4\mathbf{v}_{\lambda,\pi})$;

(ii) if $X^{k+1}_q = X^-_l(T^{k+1}_q)$ for some $l \le q$, then the zone $[\![\lfloor \mathbf{n}_\lambda X^{k+1}_q \rfloor + \mathbf{m}_\lambda, \lfloor \mathbf{n}_\lambda X^{k+1}_q \rfloor + \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}]\!]$ is completely occupied at time $\mathbf{a}_\lambda(T^{k+1}_q - 4\mathbf{v}_{\lambda,\pi})$.

Proof. We have $\tilde{H}_{T^{k+1}_q}(X^{k+1}_q) > 0$: in the limit process, a fire is stopped in X^{k+1}_q at time T^{k+1}_q by a microscopic zone. Without loss of generality, we assume that $Z_{T^{k+1}_q-}(X^{k+1}_q-) = 1$. We have either $H_{T^{k+1}_q-}(X^{k+1}_q) > 0$ or $Z_{T^{k+1}_q-}(X^{k+1}_q+) < 1$. Clearly, $X^{k+1}_q = X_m \in \mathcal{B}_M$ for some $m \le q$, with $Z_{T_m-}(X_m) < 1$ (else, we would have $H_{T^{k+1}_q}(X^{k+1}_q) = 0$ and $Z_{T^{k+1}_q-}(X^{k+1}_q-) = Z_{T^{k+1}_q-}(X^{k+1}_q+)$). We define $\mathcal{M} = (i_0; t_0, T_m)$ as in Lemma II.8.10.

By construction, there is $l \in \{1,\dots,q\}$ such that $X_m = X_l^+(T_q^{k+1})$. Hence, $T_q^{k+1} = T_l^{D,+}$ and $X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \in \chi^+_{T_q^k+4\mathbf{v}_{\lambda,\pi}}$ with $F_{T_q^k+4\mathbf{v}_{\lambda,\pi}}(y) = 0$ for all $y \in (X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi})\,, X_q^{k+1} + \alpha/p)$. By Lemma II.8.9, we deduce that there is no burning tree in $[\![\lfloor \mathbf{n}_\lambda X_l \rfloor + i^{l,+}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi}-T_l)} + 1\,, \lfloor \mathbf{n}_\lambda X_q^{k+1} \rfloor]\!]$ at time $\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi})$ whence by Lemma II.8.11, that the site $\lfloor \mathbf{n}_\lambda X_q^{k+1} \rfloor$ is not affected by a fire during $[\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi})\,, \mathbf{a}_\lambda(T_q^{k+1} - 4\mathbf{v}_{\lambda,\pi})]$. The site $\lfloor \mathbf{n}_\lambda X_q^{k+1} \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi} - 1$ is not been affected by any fire during the time interval $(\mathbf{a}_\lambda(T_q^{k+1} - 1 - 2\alpha)\,, \mathbf{a}_\lambda(T_q^{k+1} - 4\mathbf{v}_{\lambda,\pi}))$, recall Step 2 in the proof of Lemma II.8.12.

Case 1. Assume first that $H_{T_q^{k+1}-}(X_q^{k+1}) > 0$. Then by construction, there holds $T_m + Z_{T_m-}(X_m) > T_q^{k+1} > T_m$, whence by $\Omega_M(\alpha)$, $T_m + Z_{T_m-}(X_m) > T_q^{k+1} + 2\alpha > T_m + 4\alpha$.

We deduce from Lemma II.8.2 that there is a vacant site in

$$C^P = C^P((\zeta_t^{\lambda,\pi,\mathcal{M},m}(i))_{t\geq 0, i\in\mathbb{Z}}, (0, T_m)) = [\![i^g\,, i^d]\!] \subset [\![-\mathbf{m}_\lambda\,, \mathbf{m}_\lambda]\!]$$

during the time interval $[\mathbf{a}_\lambda(T_m + \kappa^0_{\lambda,\pi})\,, \mathbf{a}_\lambda(T_m + \Theta_{\mathcal{M}}^{\lambda,\pi,m})]$ (by definition of $\Theta_{\mathcal{M}}^{\lambda,\pi,m}$). By Lemma II.8.10 and with our coupling (recall that seeds fall on $(X_m)_\lambda$ according to the processes $(N_t^{S,m}(i - \lfloor \mathbf{n}_\lambda X_m \rfloor))_{t\geq 0, i\in(X_m)}$), we deduce that there is also a vacant site in $[\![\lfloor \mathbf{n}_\lambda X_m \rfloor + i^g\,, \lfloor \mathbf{n}_\lambda X_m \rfloor + i^d]\!] \subset (X_m)_\lambda$ during $[\mathbf{a}_\lambda(T_m + \kappa^0_{\lambda,\pi})\,, \mathbf{a}_\lambda(T_m + \Theta_{\mathcal{M}}^{\lambda,\pi,m})]$. But by $\Omega_4^{S,P}(\gamma, \lambda, \pi)$, we see that $\Theta_{\mathcal{M}}^{\lambda,\pi,m} \geq Z_{T_m-}(X_m) - \gamma$ whence $T_m + \Theta_{\mathcal{M}}^{\lambda,\pi,m} \geq T_m + Z_{T_m-}(X_m) - \gamma > T_q^{k+1} + 2\alpha - \gamma > T_q^{k+1} + \mathbf{v}_{\lambda,\pi}$ since $\gamma < \alpha$ and $\mathbf{v}_{\lambda,\pi} < \alpha$. All this implies that there is a vacant site in $C^P \subset (X_m)_\lambda$ during $[\mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda,\pi})\,, \mathbf{a}_\lambda(T_q^{k+1} + \mathbf{v}_{\lambda,\pi})]$.

Since the match falling on $\lfloor \mathbf{n}_\lambda X_m \rfloor$ does not affect the zone outside $(X_m)_\lambda$, we deduce from the preliminary considerations that the zone $[\![\lfloor \mathbf{n}_\lambda X_q^{k+1} \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}\,, \lfloor \mathbf{n}_\lambda X_q^{k+1} \rfloor - \mathbf{m}_\lambda]\!]$ is not affected by any fire during $[\mathbf{a}_\lambda(T_q^{k+1} - 1 - \alpha)\,, \mathbf{a}_\lambda(T_q^{k+1} - 4\mathbf{v}_{\lambda,\pi})]$, which implies the claim by $\Omega_3^S(\lambda, \pi)$.

Case 2. Assume that $H_{T_q^{k+1}-}(X_m) = 0$. Then by construction, there holds $T_q^{k+1} - [T_m - Z_{T_m-}(X_m)] \geq 1$, whence $T_q^{k+1} - [T_m - Z_{T_m-}(X_m)] \geq 1 + 3\alpha$. Since $H_{T_q^{k+1}-}(X_m) = 0$, we have $Z_{T_q^{k+1}-}(X_m+) < 1 = Z_{T_q^{k+1}-}(X_m-)$ and $T_m + Z_{T_m-}(X_m) \leq T_q^{k+1}$, so that $T_m + Z_{T_m-}(X_m) \leq T_q^{k+1} - 3\alpha$.

We aim to use the event $\Omega_1^{S,P}(\lambda, \pi, \alpha)$. We recall that $t_0 = T_m - Z_{T_m-}(X_m) = \tau_{T_m-}(X_m)$. Observe that $Z_{t_0-}(X_m) = Z_{t_0-}(X_m-) = Z_{t_0-}(X_m+) = 1$ because there is no match falling on x during $[0\,, T_m)$.

Set now $t_1 = T_m$. Observe that $0 < t_1 - t_0 < 1$ (because $Z_{T_m}(X_m) < 1$). Necessarily, $Z_{t-}(x_+)$ has jumped to 0 at least one time between t_0 and $T_q^{k+1}-$ (else, one would have $Z_{T_q^{k+1}-}(x_+) = 1$, since $T_q^{k+1} - t_0 \geq 1$ by assumption) and this jump occurs after $t_0 + 1 > t_1$ (since a jump of $Z_{t-}(x_+)$ requires that $Z_{t-}(x_+) = 1$, and since for all $t \in (t_0, t_0 + 1)$, $Z_{t-}(x_+) = t - t_0 < 1$).

We thus may denote by $t_2 < t_3 < \cdots < t_K$, for some $K \geq 2$, the successive times of jumps of the process $(Z_{t-}(x_-), Z_{t-}(x_+))$ during (t_0+1, T_q^{k+1}). Then we observe that $Z_{t-}(x+)$ and $Z_{t-}(x-)$ do never jump to 0 at the same time during $(t_0\,, T_q^{k+1})$ (else it would mean that x is crossed by a fire at some time u, whence necessarily $H_r(x) = 0$ and $Z_{r-}(x+) = Z_{r-}(x-)$ for all $r \in [u\,, T_q^{k+1}]$).

Furthermore there is always at least one jump of $(Z_{t-}(x_-), Z_{t-}(x_+))$ of any time interval of length 1 (during $(t_0\,, T_q^{k+1})$), because else, $Z_{t-}(x_-)$ and $Z_{t-}(x_+)$ would both become to be equal to 1 and thus would remain equal forever.

Finally, observe that two jumps of $Z_{t-}(x_+)$ cannot occur in a time of length 1 (since a jump of $Z_{t-}(x_+)$ requires that $Z_{t-}(x_+) = 1$) and the same thing holds for $Z_{t-}(x_-)$.

Consequently the family $\mathcal{P} = \{t_0, \ldots, t_K\}$ necessarily satisfies the condition $(PP1)$ of Subsection II.8.3.

For each $l \in \{0, 2, \ldots, K\}$, there is a unique (thanks to $\Omega_M(\alpha)$) $k_l \in [\![0\,, q]\!]$ such that $t_l = T_{k_l}(X_m)$. We set, for all $l \in \{0, 2, \ldots, K\}$,

$$i_l = \lfloor \mathbf{n}_\lambda X_{k_l} \rfloor + i^{k_l,+}_{\mathbf{a}_\lambda(t_l - \mathbf{v}_{\lambda,\pi} - T_{k_l})} - \lfloor \mathbf{n}_\lambda X_m \rfloor$$

if the jump at time t_l is a jump of $Z_{t-}(X_m-)$ (that is if $x = X^+_{k_l}(t_l)$) and

$$i_l = \lfloor \mathbf{n}_\lambda X_{k_l} \rfloor + i^{k_l,-}_{\mathbf{a}_\lambda(t_l - \mathbf{v}_{\lambda,\pi} - T_{k_l})} - \lfloor \mathbf{n}_\lambda X_m \rfloor$$

if the jump at time t_l is a jump of $Z_{t-}(X_m+)$ (that is if $x = X^-_{k_l}(t_l)$). Set for example $i_0 = 0$ if $t_0 = 0$. We also put $\varepsilon = -1$ if $x = X^+_{l_2}(t_2)$ and $\varepsilon = 1$ else. We thus may denote $\mathcal{I} = (\varepsilon; i_{k_0}, i_{k_2}, \ldots, i_{k_K})$. Clearly, $\mathcal{I}$ satisfies $(PP2)$, thanks to (II.8.25).

All this implies that $\mathfrak{P} = (\mathcal{P}, \mathcal{I})$ satisfies (PP).

Next, there holds that $t_2 - t_1 < Z_{T_m-}(X_m) = t_1 - t_0$, because else, we would have $H_{t_2-}(X_m) = 0$ and thus the fire k_2 would cross X_m, so that $Z_{t-}(x_+)$ and $Z_{t-}(x_-)$ would remain equal forever. Furthermore, we have $0 < T_q^{k+1} - t_K < 1$ because else, we would have $Z_{T_q^{k+1}}(X_m-) = Z_{T_q^{k+1}}(X_m+) = 1$, whence $T_q^{k+1} < t_K - 3\alpha$.

Finally, we check that

$$(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}(i))_{t\in[t_0-\mathbf{v}_{\lambda,\pi}, t_K+4\mathbf{v}_{\lambda,\pi}], i\in(X_m)_\lambda} = (\zeta^{\lambda,\pi,\mathfrak{P},m}_{\mathbf{a}_\lambda t}(i - \lfloor \mathbf{n}_\lambda x \rfloor))_{t\in[t_0-\mathbf{v}_{\lambda,\pi}, t_K+4\mathbf{v}_{\lambda,\pi}], i\in(X_m)_\lambda} \tag{II.8.34}$$

this last process being built with the family of seed processes $(N^{S,m}_t(i))_{t\geq 0, i\in\mathbb{Z}}$ and the family of propagation processes $(N^{P,m}_t(i))_{t\geq 0, i\in\mathbb{Z}}$ as in Subsection II.8.3. We do *e.g.* it in the case where $\varepsilon = -1$ and $t_0 > 1$, the other cases being treated similarly.

Observe that for all $l \in \{0, 2, \ldots, K\}$ there holds $t_l = T_{k_l}(X_m) = T^{D,+}_{k_l}$ (if $X_m = X^+_{k_l}(t_l)$) or $T^{D,-}_{k_l}$ (if $X_m = X^-_{k_l}(t_l)$). Hence, since $T^k_q + 4\mathbf{v}_{\lambda,\pi} \geq T_l + \mathbf{v}_{\lambda,\pi}$, we have

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda(t_l - \mathbf{v}_{\lambda,\pi})}(\lfloor \mathbf{n}_\lambda X_m \rfloor + i_l) = 2 \tag{II.8.35}$$

for all $l \in \{0, 2, \ldots, K\}$, thanks to $\Omega^{\lambda,\pi}_{T^k_q + 4\mathbf{v}_{\lambda,\pi}}$.

We already have checked in Lemma II.8.10 that $(\eta^{\lambda,\pi}_t(i))_{t\geq 0, i\in\mathbb{Z}}$ and $(\zeta^{\lambda,\pi,\mathfrak{P},m}_t(i-\lfloor \mathbf{n}_\lambda x\rfloor))_{t\geq 0,i\in\mathbb{Z}}$ are equal on $(X_m)_\lambda$ during the time interval $[\mathbf{a}_\lambda(t_0-\mathbf{v}_{\lambda,\pi})\,,\mathbf{a}_\lambda(T_m+\kappa^0_{\lambda,\pi})]$. Using similar argument, observing that seeds fall on $[X_m]_{\lambda,\pi}$ and fires spreads through $[X_m]_{\lambda,\pi}$ according to the same processes and using (II.8.35), we easily deduce that (II.8.34) holds on $\Omega(\alpha,\gamma,\lambda,\pi)$.

We thus can use $\Omega^{S,P}_1(\lambda,\pi,\alpha)$ and conclude that

- there is $i\in(X_m)_\lambda$ with $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}(i)=0$ for all $t\in[T^{k+1}_q-\mathbf{v}_{\lambda,\pi}\,,T^{k+1}_q+\mathbf{v}_{\lambda,\pi}]\subset[t_K+2\mathbf{v}_{\lambda,\pi}\,,t_K+1-\alpha]$;

- no fire coming from the right can affect the zone on the left of $\lfloor \mathbf{n}_\lambda X^{k+1}_q\rfloor-\mathbf{m}_\lambda$ during the time interval $[\mathbf{a}_\lambda T_m\,,\mathbf{a}_\lambda(T^{k+1}_q-4\mathbf{v}_{\lambda,\pi})]$ (because the fire are stopped by vacant site in $(X_m)_\lambda$). Hence, to affect the zone $[\![\lfloor \mathbf{n}_\lambda X^{k+1}_q\rfloor-\mathbf{m}_\lambda-2\mathbf{k}_{\lambda,\pi}\,,\lfloor \mathbf{n}_\lambda X^{k+1}_q\rfloor-\mathbf{m}_\lambda]\!]$ during this time interval, a fire must come from the left and thus must affect the site $\lfloor \mathbf{n}_\lambda X^{k+1}_q\rfloor-\mathbf{m}_\lambda-2\mathbf{k}_{\lambda,\pi}-1$. We deduce from the preliminary considerations that the zone $[\![\lfloor \mathbf{n}_\lambda X^{k+1}_q\rfloor-\mathbf{m}_\lambda-2\mathbf{k}_{\lambda,\pi}\,,\lfloor \mathbf{n}_\lambda X^{k+1}_q\rfloor-\mathbf{m}_\lambda]\!]$ is not affected by any fire during $[\mathbf{a}_\lambda(T^{k+1}_q-1-\alpha)\,,\mathbf{a}_\lambda(T^{k+1}_q-4\mathbf{v}_{\lambda,\pi})]$ which implies the claim by $\Omega^S_3(\lambda,\pi)$. ☐

We deduce the following corollary, which is the goal of Stage 2.

Corollary II.8.14. *On $\Omega(\alpha,\gamma,\lambda,\pi)$, $\Omega^{\lambda,\pi}_{T^k_q+4\mathbf{v}_{\lambda,\pi}}$ implies $\Omega^{\lambda,\pi}_{T^{k+1}_q+4\mathbf{v}_{\lambda,\pi}}$.*

Proof. We have to prove that for $l\leq q$,

(a) if $X^+_l(T^k_q+4\mathbf{v}_{\lambda,\pi})\in\chi^+_{T^k_q+4\mathbf{v}_{\lambda,\pi}}$ and if $T^{D,+}_l\neq T^{k+1}_q$, then $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(\lfloor \mathbf{n}_\lambda X_l\rfloor+i^{l,+}_{\mathbf{a}_\lambda(s-T_l)})=2$ for all $s\in[T^k_q+4\mathbf{v}_{\lambda,\pi}\,,T^{k+1}_q+4\mathbf{v}_{\lambda,\pi}]$;

(b) if $X^-_l(T^k_q+4\mathbf{v}_{\lambda,\pi})\in\chi^-_{T^k_q+4\mathbf{v}_{\lambda,\pi}}$ and if $T^{D,-}_l\neq T^{k+1}_q$, then $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(\lfloor \mathbf{n}_\lambda X_l\rfloor+i^{l,-}_{\mathbf{a}_\lambda(s-T_l)})=2$ for all $s\in[T^k_q+4\mathbf{v}_{\lambda,\pi}\,,T^{k+1}_q+4\mathbf{v}_{\lambda,\pi}]$;

(c) if $X^+_l(T^k_q+4\mathbf{v}_{\lambda,\pi})\in\chi^+_{T^k_q+4\mathbf{v}_{\lambda,\pi}}$ and if $T^{D,+}_l=T^{k+1}_q$, then $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(\lfloor \mathbf{n}_\lambda X_l\rfloor+i^{l,+}_{\mathbf{a}_\lambda(s-T_l)})=2$ for all $s\in[T^k_q+4\mathbf{v}_{\lambda,\pi}\,,T^{k+1}_q-\mathbf{v}_{\lambda,\pi}]$ and there is $s\in[T^{k+1}_q-\mathbf{v}_{\lambda,\pi}\,,T^{k+1}_q+\mathbf{v}_{\lambda,\pi}]$ such that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(\lfloor \mathbf{n}_\lambda X_l\rfloor+i^{l,+}_{\mathbf{a}_\lambda(s-T_l)})=0$;

(d) if $X^-_l(T^k_q+4\mathbf{v}_{\lambda,\pi})\in\chi^-_{T^k_q+4\mathbf{v}_{\lambda,\pi}}$ and if $T^{D,-}_l=T^{k+1}_q$, then $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(\lfloor \mathbf{n}_\lambda X_l\rfloor+i^{l,-}_{\mathbf{a}_\lambda(s-T_l)})=2$ for all $s\in[T^k_q+4\mathbf{v}_{\lambda,\pi}\,,T^{k+1}_q-\mathbf{v}_{\lambda,\pi}]$ and there is $s\in[T^{k+1}_q-\mathbf{v}_{\lambda,\pi}\,,T^{k+1}_q+\mathbf{v}_{\lambda,\pi}]$ such that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(\lfloor \mathbf{n}_\lambda X_l\rfloor+i^{l,-}_{\mathbf{a}_\lambda(s-T_l)})=0$.

All this will imply the result (observe that only these four cases may occur).

Observe that either $F_{T^{k+1}_q}(X^{k+1}_q)=2$ (*i.e.* two fires meet at time T^{k+1}_q) or $F_{T^{k+1}_q}(X^{k+1}_q)\leq 1$ (*i.e.* a fire is stopped by a microscopic zone).

Step 1. We start by studying the case where $F_{T_q^{k+1}}(X_q^{k+1}) = 2$. There are l_1 and l_2 such that $X_{l_1}^+(T_q^{k+1}) = X_q^{k+1} = X_{l_2}^-(T_q^{k+1})$. In this Step, we prove (c) for the fire l_1 and (d) for the fire l_2.

By construction, we have $X_{l_1}^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \in \chi^+_{T_q^k+4\mathbf{v}_{\lambda,\pi}}$ and $X_{l_2}^-(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \in \chi^-_{T_q^k+4\mathbf{v}_{\lambda,\pi}}$ with $F_{T_q^k+4\mathbf{v}_{\lambda,\pi}}(y) = 0$ for all $y \in (X_{l_1}^+(T_q^k+4\mathbf{v}_{\lambda,\pi}), X_{l_2}^-(T_q^k+4\mathbf{v}_{\lambda,\pi}))$ and $X_{l_2}^-(T_q^k+4\mathbf{v}_{\lambda,\pi}) - X_{l_1}^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) = 2(T_q^{k+1} - T_q^k - 4\mathbf{v}_{\lambda,\pi})/p \geq 5\alpha/p$.

We first prove that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(\lfloor \mathbf{n}_\lambda X_{l_1} \rfloor + i^{l_1,+}_{\mathbf{a}_\lambda(s-T_{l_1})}) = 2$ for all $s \in [\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda,\pi})]$. Equivalently, we prove that

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_{l_1} + T^{l_1}_{j-\lfloor \mathbf{n}_\lambda X_{l_1}\rfloor}}(j) = 2$$

for all $j \in [\![\lfloor \mathbf{n}_\lambda X_{l_1} \rfloor + i^{l_1,+}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi}-T_{l_1})}, \lfloor \mathbf{n}_\lambda X_{l_1} \rfloor + i^{l_1,+}_{\mathbf{a}_\lambda(T_q^{k+1}-\mathbf{v}_{\lambda,\pi}-T_{l_1})}]\!]$.

Firstly, Lemma II.8.12 with $s_0 = T_q^{k+1}$ directly implies that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_{l_1} + T^{l_1}_{j-\lfloor \mathbf{n}_\lambda X_{l_1}\rfloor}}(j) = 2$ for all $j \in [\![\lfloor \mathbf{n}_\lambda X_{l_1} \rfloor + i^{l_1,+}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi}-T_{l_1})}, \lfloor \mathbf{n}_\lambda X_q^{k+1} \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}]\!]$.

Secondly, we prove that

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda(T_q^{k+1}-4\mathbf{v}_{\lambda,\pi})}(i) = 1 \text{ for all } i \in [X_q^{k+1}]_{\lambda,\pi}.$$

This will completes the claim, using similar arguments as in Lemma II.8.12 since there is no burning tree in $[\![\lfloor \mathbf{n}_\lambda X_{l_1} \rfloor + i^{l_1,+}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi}-T_{l_1})} + 1, \lfloor \mathbf{n}_\lambda X_{l_2} \rfloor + i^{l_2,-}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi}-T_{l_2})} + 1]\!]$ at time $\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi})$ (by Lemma II.8.9) and since $\lfloor \mathbf{n}_\lambda X_{l_1} \rfloor + i^{l_1,+}_{\mathbf{a}_\lambda(T_q^{k+1}-\mathbf{v}_{\lambda,\pi}-T_{l_1})} \leq \lfloor \mathbf{n}_\lambda X_q^{k+1} \rfloor - \mathbf{m}_\lambda$ and $\lfloor \mathbf{n}_\lambda X_{l_2} \rfloor + i^{l_2,-}_{\mathbf{a}_\lambda(T_q^{k+1}-\mathbf{v}_{\lambda,\pi}-T_{l_2})} \geq \lfloor \mathbf{n}_\lambda X_q^{k+1} \rfloor + \mathbf{m}_\lambda$ (by $\Omega^{P,T}(\lambda,\pi)$ and (II.8.25)).

No fire can affect the zone $[X_q^{k+1}]_{\lambda,\pi}$ during $[\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(T_q^{k+1} - 4\mathbf{v}_{\lambda,\pi})]$, thanks to (II.8.28) and to Lemma II.8.9, (which implies that there is no burning tree in $[\![\lfloor \mathbf{n}_\lambda X_{l_1} \rfloor + i^{l_1,+}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi}-T_{l_1})} + 1, \lfloor \mathbf{n}_\lambda X_{l_2} \rfloor + i^{l_2,-}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi}-T_{l_2})} - 1]\!]$). By construction, we have $Z_{T_q^{k+1}-}(X_q^{k+1}) = Z_{T_q^{k+1}-}(X_q^{k+1}+) = Z_{T_q^{k+1}-}(X_q^{k+1}-) = 1$, whence $T_q^{k+1} - \tau_{T_q^{k+1}}(X_q^{k+1}) \geq 1$ and $T_q^{k+1} - \tau_{T_q^{k+1}}(X_q^{k+1}) \geq 1 + 3\alpha$ by $\Omega_M(\alpha)$. Since no match has fallen on $X_q^{k+1} \in \mathcal{B}_M^2$ during $[0, T_q^{k+1}]$, using similar argument as in Case 1 Step 3 in the proof of Lemma II.8.12, we then deduce that for all $j \in [X_q^{k+1}]_{\lambda,\pi}$,

$$\rho^{\lambda,\pi}_{\mathbf{a}_\lambda(T_q^{k+1}-4\mathbf{v}_{\lambda,\pi})}(j) \leq T_q^{k+1} - 1 - \alpha,$$

which implies the claim by $\Omega_3^S(\lambda,\pi)$. Same thing of course holds for l_2.

Furthermore, we have shown that at time $\mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda,\pi})$, the sites $\lfloor \mathbf{n}_\lambda X_{l_1} \rfloor + i^{l_1,+}_{\mathbf{a}_\lambda(T_q^{k+1}-\mathbf{v}_{\lambda,\pi}-T_{l_1})}$ and $\lfloor \mathbf{n}_\lambda X_{l_2} \rfloor + i^{l_2,-}_{\mathbf{a}_\lambda(T_q^{k+1}-\mathbf{v}_{\lambda,\pi}-T_{l_2})}$ are burning and

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda(T_q^{k+1}-\mathbf{v}_{\lambda,\pi})}(i) = 1 \tag{II.8.36}$$

for all $i \in [\![\lfloor \mathbf{n}_\lambda X_{l_1} \rfloor + i^{l_1,+}_{\mathbf{a}_\lambda(T_q^{k+1}-\mathbf{v}_{\lambda,\pi}-T_{l_1})} + 1\,, \lfloor \mathbf{n}_\lambda X_{l_2} \rfloor + i^{l_2,-}_{\mathbf{a}_\lambda(T_q^{k+1}-\mathbf{v}_{\lambda,\pi}-T_{l_2})} - 1]\!]$.

We next show that the fires are stopped during $[\mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda,\pi})\,, \mathbf{a}_\lambda(T_q^{k+1} + \mathbf{v}_{\lambda,\pi})]$. Observe that, on $\Omega^{P,T}(\lambda,\pi)$, thanks to (II.8.26), there is $i_0 \in [X_q^{k+1}]_{\lambda,\pi}$ such that

$$i_0 = \lfloor \mathbf{n}_\lambda X_{l_1} \rfloor + i^{l_1,+}_{T^{l_1}_{i_0+1-\lfloor \mathbf{n}_\lambda X_{l_1} \rfloor -}} = \lfloor \mathbf{n}_\lambda X_{l_2} \rfloor + i^{l_2,-}_{T^{l_2}_{i_0-1-\lfloor \mathbf{n}_\lambda X_{l_2} \rfloor -}}.$$

We deduce from (II.8.36), that

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_{l_1} + T^{l_1}_{j-\lfloor \mathbf{n}_\lambda X_{l_1} \rfloor}}(j) = 2 \text{ for all } j \in [\![\lfloor \mathbf{n}_\lambda X_{l_1} \rfloor + i^{l_1,+}_{\mathbf{a}_\lambda(T_q^{k+1}-\mathbf{v}_{\lambda,\pi}-T_{l_1})}\,, i_0]\!]$$

and

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_{l_2} + T^{l_2}_{j-\lfloor \mathbf{n}_\lambda X_{l_2} \rfloor}}(j) = 2 \text{ for all } j \in [\![i_0\,, \lfloor \mathbf{n}_\lambda X_{l_2} \rfloor + i^{l_2,-}_{\mathbf{a}_\lambda(T_q^{k+1}-\mathbf{v}_{\lambda,\pi}-T_{l_2})}]\!].$$

We know that the fire in i_0 propagates at time

$$\mathbf{a}_\lambda T_{l_1} + T^{l_1}_{i_0+1-\lfloor \mathbf{n}_\lambda X_{l_1} \rfloor} = \mathbf{a}_\lambda T_{l_2} + T^{l_2}_{i_0-1-\lfloor \mathbf{n}_\lambda X_{l_2} \rfloor}.$$

Thus, on $\Omega^{P,T}(\lambda,\pi)$, with our coupling, at time $\mathbf{a}_\lambda T_{l_1} + T^{l_1}_{i_0+1-\lfloor \mathbf{n}_\lambda X_{l_1} \rfloor}$, either the site i_0+1 is vacant (because it has been burnt by the fire l_2) or the site i_0+1 is occupied but has vacant neighbors until it propagates, that is until $\mathbf{a}_\lambda T_{l_1} + T^{l_1}_{i_0+2-\lfloor \mathbf{n}_\lambda X_{l_1} \rfloor}$ (because it is a spark for the fire l_2). In any case, since

$$\mathbf{a}_\lambda T_{l_1} + T^{l_1}_{i_0+2-\lfloor \mathbf{n}_\lambda X_{l_1} \rfloor} \in [\mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda,\pi})\,, \mathbf{a}_\lambda(T_q^{k+1} + \mathbf{v}_{\lambda,\pi})],$$

recall (II.8.30), there is $s_1 \in [T_q^{k+1} - \mathbf{v}_{\lambda,\pi}\,, T_q^{k+1} + \mathbf{v}_{\lambda,\pi}]$ such that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(\lfloor \mathbf{n}_\lambda X_{l_1} \rfloor + i^{l_1,+}_{\mathbf{a}_\lambda(s_1-T_{l_1})}) = 0$. Similarly, we can find $s_2 \in [T_q^{k+1} - \mathbf{v}_{\lambda,\pi}\,, T_q^{k+1} + \mathbf{v}_{\lambda,\pi}]$ such that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s_2}(\lfloor \mathbf{n}_\lambda X_{l_2} \rfloor + i^{l_2,+}_{\mathbf{a}_\lambda(s_2-T_{l_2})}) = 0$, which completes this Step.

Step 2. Here, we study the case where $F_{T_q^{k+1}}(X_q^{k+1}) \leq 1$ and $X_q^{k+1} \notin \{-A, A\}$. Assume for example that $X_q^{k+1} = X^+_{l_0}(T_q^{k+1})$ for some $l_0 \leq q$. In this Step, we prove (c) for the fire l_0.

By construction, $X^+_{l_0}(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \in \chi^+_{T_q^k+4\mathbf{v}_{\lambda,\pi}}$ and $F_{T_q^k+4\mathbf{v}_{\lambda,\pi}}(y) = 0$ for all $y \in (X^+_{l_0}(T_q^k + 4\mathbf{v}_{\lambda,\pi})\,, X_q^{k+1} + \alpha/p)$.

We first prove that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(\lfloor \mathbf{n}_\lambda X_{l_0} \rfloor + i^{l_0,+}_{\mathbf{a}_\lambda(s-T_{l_0})}) = 2$ for all $s \in [\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi})\,, \mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda,\pi})]$. Equivalently, we prove that

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_{l_0} + T^{l_0}_{j-\lfloor \mathbf{n}_\lambda X_{l_0} \rfloor}}(j) = 2$$

for all $j \in [\![\lfloor \mathbf{n}_\lambda X_{l_0} \rfloor + i^{l_0,+}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi}-T_{l_0})}\,, \lfloor \mathbf{n}_\lambda X_{l_0} \rfloor + i^{l_0,+}_{\mathbf{a}_\lambda(T_q^{k+1}-\mathbf{v}_{\lambda,\pi}-T_{l_0})}]\!]$.

Firstly, using Lemma II.8.12 with $s_0 = T_q^{k+1}$, we deduce that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_{l_0} + T^{l_0}_{j-\lfloor \mathbf{n}_\lambda X_{l_0}\rfloor}}(j) = 2$ for all $j \in [\![\lfloor \mathbf{n}_\lambda X_{l_0}\rfloor + i^{l_0,+}_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi} - T_{l_0})}\,,\lfloor \mathbf{n}_\lambda X_q^{k+1}\rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}]\!]$.

Secondly, Lemma II.8.13-1 shows that the zone $[\![\lfloor \mathbf{n}_\lambda X_q^{k+1}\rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}\,, \lfloor \mathbf{n}_\lambda X_q^{k+1}\rfloor - \mathbf{m}_\lambda]\!]$ is completely occupied at time $\mathbf{a}_\lambda(T_q^{k+1} - 4\mathbf{v}_{\lambda,\pi})$. Since no fire coming from the right can affect the zone on the left of $\lfloor \mathbf{n}_\lambda X_q^{k+1}\rfloor$ until $\mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda,\pi})$, we deduce the claim using similar argument as in Lemma II.8.12.

Finally, Lemma II.8.13 directly imply that there is $j \in (X_q^{k+1})_\lambda$ such that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(j) = 0$ for all $s \in [T_q^{k+1} - \mathbf{v}_{\lambda,\pi}\,, T_q^{k+1} + \mathbf{v}_{\lambda,\pi}]$. Since

$$\lfloor \mathbf{n}_\lambda X_{l_0}\rfloor + i^{l_0,+}_{\mathbf{a}_\lambda(T_q^{k+1} + \mathbf{v}_{\lambda,\pi} - T_{l_0})} \geq \lfloor \mathbf{n}_\lambda X_q^{k+1}\rfloor + \mathbf{m}_\lambda,$$

recall (II.8.26), there is $s \in [T_q^{k+1} - \mathbf{v}_{\lambda,\pi}\,, T_q^{k+1} + \mathbf{v}_{\lambda,\pi}]$ such that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(\lfloor \mathbf{n}_\lambda X_{l_0}\rfloor + i^{l_0,+}_{\mathbf{a}_\lambda(s - T_{l_0})}) = 0$, as desired.

Step 3. Here we study the case where $X_q^{k+1} \in \{-A, A\}$. Assume for example that $X_q^{k+1} = X^+_{l_0}(T_q^{k+1}) = A$ for some $l_0 \leq q$. In this Step, we prove (c) for the fire l_0.

This case is very simple: by construction, $X^+_{l_0}(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \in \chi^+_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}$ and $F_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}(y) = 0$ for all $y \in (X^+_{l_0}(T_q^k + 4\mathbf{v}_{\lambda,\pi})\,, A)$.

Since there is no burning tree in $[\![\lfloor \mathbf{n}_\lambda X_{l_0}\rfloor + i^{l_0,+}_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi} - T_{l_0})} + 1\,, \lfloor \mathbf{n}_\lambda A\rfloor]\!]$ at time $\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi})$ (thanks to Lemma II.8.9), we deduce, using similar argument as in the proof of Lemma II.8.12, that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_{l_0} + T^l_{j-\lfloor \mathbf{n}_\lambda X_l\rfloor}}(j) = 2$ for all $j \in [\![\lfloor \mathbf{n}_\lambda X_{l_0}\rfloor + i^{l_0,+}_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi} - T_{l_0})}\,, \lfloor \mathbf{n}_\lambda A\rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}]\!]$. The zone $[\![\lfloor \mathbf{n}_\lambda A\rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}\,, \lfloor \mathbf{n}_\lambda A\rfloor]\!]$ is not affected by any fire during $[\mathbf{a}_\lambda(T_q^{k+1} - 1 - \alpha)\,, \mathbf{a}_\lambda(T_q^{k+1} - 4\mathbf{v}_{\lambda,\pi})]$ (recall Step 3 in the proof of Lemma II.8.12) and no match falls in this zone during $[0\,, \mathbf{a}_\lambda T]$. We deduce as usual, using $\Omega_3^S(\lambda, \pi)$, that this zone is completely occupied at time $\mathbf{a}_\lambda(T_q^{k+1} - 4\mathbf{v}_{\lambda,\pi})$. Thus, we have

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_{l_0} + T^l_{j-\lfloor \mathbf{n}_\lambda X_{l_0}\rfloor}}(j) = 2$$

for all $j \in [\![\lfloor \mathbf{n}_\lambda A\rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}\,, \lfloor \mathbf{n}_\lambda A\rfloor]\!]$, which implies the claim since $\lfloor \mathbf{n}_\lambda X_{l_0}\rfloor + i^{l_0,+}_{\mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda,\pi} - T_{l_0})} \leq \lfloor \mathbf{n}_\lambda A\rfloor - \mathbf{m}_\lambda$.

We immediately deduce the claim since $\eta^{\lambda,\pi}_s(\lfloor \mathbf{n}_\lambda A\rfloor + 1) = 0$ for all $s \in [0\,, \infty)$.

Step 4. Here we study the case where $x_0 := X^+_{l_0}(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \in \chi^+_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}$ with $T^{D,+}_{l_0} \neq T_q^{k+1}$, for some $l_0 \leq q$. We prove (a) for the fire l_0.

By $\Omega_M(\alpha)$, we have $T^{D,+}_{l_0} \geq T_q^{k+1} + 3\alpha$. If $F_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}(y) = 0$ for all $y > x_0$, necessarily $F_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}(y) = 0$ for all $y \in (x_0\,, X^+_{l_0}(T_q^{k+1} + 3\alpha))$. Lemma II.8.12 with $s_0 = T_q^{k+1} + 2\alpha$

directly implies the result, since on $\Omega^{P,T}(\lambda,\pi)$, recall (II.8.24), there holds that

$$\begin{aligned}\lfloor \mathbf{n}_\lambda X_{l_0}\rfloor + i^{l_0,+}_{\mathbf{a}_\lambda(T_q^{k+1}+4\mathbf{v}_{\lambda,\pi}-T_{l_0})} &\le \lfloor \mathbf{n}_\lambda X^+_{l_0}(T_q^{k+1}+4\mathbf{v}_{\lambda,\pi})\rfloor + \mathbf{k}_{\lambda,\pi}\\ &\le \lfloor \mathbf{n}_\lambda X^+_{l_0}(T_q^{k+1}+2\alpha)\rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}.\end{aligned}$$

Else, we define

$$x_1 := \inf\left\{y > x_0 : F_{T_q^k+4\mathbf{v}_{\lambda,\pi}}(y) = 1\right\}$$

and distinguish several cases.

Case 1. Assume that $x_1 - x_0 > (T_q^{k+1} - T_q^k + 2\alpha)/p$. Using Lemma II.8.12 with $s_0 = T_q^{k+1} + \alpha$, we immediately deduce that

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_{l_0} + T^l_{i-\lfloor \mathbf{n}_\lambda X_{l_0}\rfloor}}(i) = 2$$

for all $i \in [\![\lfloor \mathbf{n}_\lambda X_l\rfloor + i^{l_0,+}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi}-T_{l_0})}\,, \lfloor \mathbf{n}_\lambda X^+_{l_0}(T_q^{k+1}+\alpha)\rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}]\!]$ whence

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(\lfloor \mathbf{n}_\lambda X_{l_0}\rfloor + i^{l_0,+}_{\mathbf{a}_\lambda(s-T_{l_0})}) = 2 \text{ for all } s \in [T_q^k + 4\mathbf{v}_{\lambda,\pi}\,, T_q^{k+1} + 4\mathbf{v}_{\lambda,\pi}]$$

because on $\Omega^{P,T}(\lambda,\pi)$, there holds $\lfloor \mathbf{n}_\lambda X_{l_0}\rfloor + i^{l_0,+}_{\mathbf{a}_\lambda(T_q^{k+1}+4\mathbf{v}_{\lambda,\pi}-T_{l_0})} \le \lfloor \mathbf{n}_\lambda X^+_l(T_q^{k+1}+\alpha)\rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}$.

Case 2. Assume that $x_1 - x_0 \le (T_q^{k+1} - T_q^k + 2\alpha)/p$ but $F_{T_q^k+4\mathbf{v}_{\lambda,\pi}}(y) = 0$ for all $y \in (x_1\,, x_1 + (T_q^{k+1} - T_q^k + 2\alpha)/p)$. Necessarily $x_1 = X^+_{l_1}(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \in \chi^+_{T_q^k+4\mathbf{v}_{\lambda,\pi}}$ for some $l_1 \le q$.

Using Lemma II.8.12 with $s_0 = T_q^{k+1} \le T^{D,+}_{l_1}$, we deduce that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_{l_1} + T^{l_1}_{i-\lfloor \mathbf{n}_\lambda X_{l_1}\rfloor}}(i) = 2$ for all $i \in [\![\lfloor \mathbf{n}_\lambda X_{l_1}\rfloor + i^{l_1,+}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi}-T_{l_1})}\,, \lfloor \mathbf{n}_\lambda X^+_{l_1}(T_q^{k+1})\rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}]\!]$. Thus, using (II.8.28), we deduce

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}\left(\lfloor \mathbf{n}_\lambda X_{l_1}\rfloor + i^{l_1,+}_{\mathbf{a}_\lambda(s-T_{l_1})}\right) = 2 \text{ for all } s \in [T_{l_1}\,, T_q^{k+1} - 4\mathbf{v}_{\lambda,\pi}].$$

We now prove that for all $i \in [\![\lfloor \mathbf{n}_\lambda X_{l_0}\rfloor + i^{l_0,+}_{\mathbf{a}_\lambda(T_q^k+4\mathbf{v}_{\lambda,\pi}-T_{l_0})}\,, \lfloor \mathbf{n}_\lambda X_{l_0}\rfloor + i^{l_0,+}_{\mathbf{a}_\lambda(T_q^{k+1}+4\mathbf{v}_{\lambda,\pi}-T_{l_0})}]\!]$, we have

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_{l_0} + T^{l_0}_{i-\lfloor \mathbf{n}_\lambda X_0\rfloor}}(i) = 2.$$

This will concludes this case.

Firstly, by construction, we have $x_1 > x_0 + 1/p$ whence by $\Omega_M(\alpha)$, $x_1 > x_0 + (1+3\alpha)/p$. Thus, using again Lemma II.8.12 with $s_0 = T_{l_0}(x_1) - \alpha$, we deduce that

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_{l_0} + T^{l_0}_{j-\lfloor \mathbf{n}_\lambda X_{l_0}\rfloor}}(j) = 2$$

for all $j \in [\![\lfloor \mathbf{n}_\lambda X_{l_0} \rfloor + i^{l_0,+}_{\mathbf{a}_\lambda(T^k_q+4\mathbf{v}_{\lambda,\pi}-T_{l_0})}, \lfloor \mathbf{n}_\lambda(x_1 - \alpha/p)\rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}]\!]$ (recall that $X^+_{l_0}(T_{l_0}(x_1)) = x_1$).

Secondly, oberve that $T_{l_1} < T^k_q$ (because else $T_{l_1} = T^k_q$ and $X^-_{l_1}(T^k_q+4\mathbf{v}_{\lambda,\pi}) \in \chi^-_{T^k_q+4\mathbf{v}_{\lambda,\pi}}$ with $x_0 < X^-_{l_1}(T^k_q + 4\mathbf{v}_{\lambda,\pi}) < X^+_{l_1}(T^k_q + 4\mathbf{v}_{\lambda,\pi}))$ whence by $\Omega_M(\alpha)$, $T_{l_1} < T^k_q - 3\alpha$. This especially imply that $T_{l_0}(y) \geq T_{l_1}(y)+1+3\alpha$ for all $y \in [x_1-3\alpha/p\,, X^+_{l_0}(T^{k+1}_q+\alpha)]$. Recall that no match falls on any site $y \in (x_1 - 3\alpha/p\,, X^+_{l_0}(T^{k+1}_q + \alpha))$ during the time interval $(T^k_q - 3\alpha\,, T^{k+1}_q + \alpha)$. Thus, in the limit process, for all $y \in (x_1 - 3\alpha/p\,, X^+_{l_0}(T^{k+1}_q + \alpha))$, we have $\tau_{T_{l_0}(y)-}(y) = T_{l_1}(y)$.

Let now $i \in [\![\lfloor \mathbf{n}_\lambda(x_1 - 2\alpha/p)\rfloor\,, \lfloor \mathbf{n}_\lambda X^+_{l_0}(T^{k+1}_q + \alpha)\rfloor]\!]$. Observe that there is no burning tree in $[\![\lfloor \mathbf{n}_\lambda X_{l_0}\rfloor + i^{l_0,+}_{\mathbf{a}_\lambda(T^k_q+4\mathbf{v}_{\lambda,\pi}-T_{l_0})} + 1\,, \lfloor \mathbf{n}_\lambda x_1\rfloor - \mathbf{k}_{\lambda,\pi}]\!]$ at time $\mathbf{a}_\lambda(T^k_q + 4\mathbf{v}_{\lambda,\pi})$, thanks to Lemma II.8.9. Since no match falls on i during $[\mathbf{a}_\lambda(T_{l_1}(i/\mathbf{n}_\lambda) + \mathbf{e}_{\lambda,\pi})\,, \mathbf{a}_\lambda(T^{k+1}_q + \alpha)]$, we deduce that no fire at all can affect the site i during the time interval $[\mathbf{a}_\lambda(T_{l_1}(i/\mathbf{n}_\lambda) + \mathbf{e}_{\lambda,\pi})\,, \mathbf{a}_\lambda T_{l_0} + T^{l_0}_{j-\lfloor \mathbf{n}_\lambda X_{l_0}\rfloor})$ whence

$$\rho^{\lambda,\pi}_{T_{l_0}+T^{l_0}_{j-\lfloor \mathbf{n}_\lambda X_{l_0}\rfloor}/\mathbf{a}_\lambda-}(i) \leq T_{l_1}(i/\mathbf{n}_\lambda) + \mathbf{e}_{\lambda,\pi}.$$

Thus, for all $i \in [\![\lfloor \mathbf{n}_\lambda(x_1 - 2\alpha/p)\rfloor\,, \lfloor \mathbf{n}_\lambda X^+_{l_0}(T^{k+1}_q + \alpha)\rfloor]\!]$, we have

$$\rho^{\lambda,\pi}_{T_{l_0}+T^{l_0}_{j-\lfloor \mathbf{n}_\lambda X_{l_0}\rfloor}/\mathbf{a}_\lambda-}(i) \leq T_{l_0}(i/\mathbf{n}_\lambda) - 1 - 3\alpha + \mathbf{e}_{\lambda,\pi}$$

and conclude using $\Omega^S_3(\lambda,\pi)$ that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_{l_0}+T^{l_0}_{i-\lfloor \mathbf{n}_\lambda X_{l_0}\rfloor}-}(i) = 1$ whence

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_{l_0}+T^{l_0}_{i-\lfloor \mathbf{n}_\lambda X_{l_0}\rfloor}}(i) = 2$$

because $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda(T^k_q+4\mathbf{v}_{\lambda,\pi})}(\lfloor \mathbf{n}_\lambda X_{l_0}\rfloor + i^{l_0,+}_{\mathbf{a}_\lambda(T^k_q+4\mathbf{v}_{\lambda,\pi}-T_{l_1})}) = 2$.

All this implies that, for all $i \in [\![\lfloor \mathbf{n}_\lambda X_{l_0}\rfloor + i^{l_0,+}_{\mathbf{a}_\lambda(T^k_q+4\mathbf{v}_{\lambda,\pi}-T_{l_0})}\,, \lfloor \mathbf{n}_\lambda X^+_{l_0}(T^{k+1}_q + \alpha)\rfloor]\!]$, we have $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_{l_0}+T^{l_0}_{i-\lfloor \mathbf{n}_\lambda X_{l_0}\rfloor}}(i) = 2$ whence

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(\lfloor \mathbf{n}_\lambda X_{l_0}\rfloor + i^{l_0,+}_{\mathbf{a}_\lambda(s-T_{l_0})}) = 2 \text{ for all } s \in [T^k_q + 4\mathbf{v}_{\lambda,\pi}\,, T^{k+1}_q + 4\mathbf{v}_{\lambda,\pi}]$$

since $\lfloor \mathbf{n}_\lambda X_{l_0}\rfloor + i^{l_0,+}_{\mathbf{a}_\lambda(T^{k+1}_q+4\mathbf{v}_{\lambda,\pi}-T_{l_0})} \leq \lfloor \mathbf{n}_\lambda X^+_{l_0}(T^{k+1}_q + \alpha)\rfloor$. This completes this case.

Case 3. In the general case, by construction, there are $x_0 < x_1 < x_2 < \cdots < x_m$ such that, for all $j \in \{0, \dots, m-1\}$,

$$x_j - x_{j+1} \leq (T^{k+1}_q - T^k_q + 2\alpha)/p$$

and

$$F_{T^k_q+4\mathbf{v}_{\lambda,\pi}}(y) = 0 \text{ for all } y \in (x_j\,, x_{j+1})$$

and finally

$$F_{T_q^k+4\mathbf{v}_{\lambda,\pi}}(y) = 0 \text{ for all } y \in (x_m\,, x_m + (T_q^{k+1} - T_q^k - 2\alpha)/p).$$

Clearly, for all $j \in \{1, \dots, m\}$, we have $x_j \in \chi^+_{T_q^k+4\mathbf{v}_{\lambda,\pi}}$ whence there exists $l_j \in \{1, \dots, q\}$ such that $x_j := X^+_{l_j}(T_q^k + \mathbf{v}_{\lambda,\pi})$.

We first prove, exactly as in Case 2, that

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(\lfloor \mathbf{n}_\lambda X_{l_m} \rfloor + i^{l_m,+}_{\mathbf{a}_\lambda(s-T_{l_m})}) = 2 \text{ for all } s \in [T_q^k + 4\mathbf{v}_{\lambda,\pi}\,, T_q^{k+1} - 4\mathbf{v}_{\lambda,\pi}].$$

Next, exactly as in Case 2, we can prove that

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(\lfloor \mathbf{n}_\lambda X_{l_{m-1}} \rfloor + i^{l_{m-1},+}_{\mathbf{a}_\lambda(s-T_{l_{m-1}})}) = 2 \text{ for all } s \in [T_q^k + 4\mathbf{v}_{\lambda,\pi}\,, T_q^{k+1} + 4\mathbf{v}_{\lambda,\pi}]$$

and so on.

Step 5. Finally, if $x_0 := X^-_{l_0}(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \in \chi^-_{T_q^k+4\mathbf{v}_{\lambda,\pi}}$ with $T^{D,+}_{l_0} \neq T_q^{k+1}$, for some $l_0 \le q$, we deduce (b) for the fire l_0 using similar argument as in Step 4.

This completes the proof. □

STAGE 3.

In this Stage, we treat the time interval $[T_q^{N_q} + 4\mathbf{v}_{\lambda,\pi}\,, T_{q+1}]$. On this time interval, no fire is stopped in the limit process. A match falls in X_{q+1} at time T_{q+1}. The proof of the following lemma is very similar to the proof of the previous Stage.

Lemma II.8.15. *On $\Omega(\alpha, \lambda, \gamma, \pi)$, $\Omega^{\lambda,\pi}_{T_q^{N_q}+4\mathbf{v}_{\lambda,\pi}}$ implies $\Omega^{\lambda,\pi}_{T_{q+1}}$.*

Sketch of the proof. Observe that $\mathcal{T}^D_M \cap (T_q^{N_q}\,, T_{q+1}) = \emptyset$. Hence, we have to prove that if $x := X^+_l(T_q^{N_q} + 4\mathbf{v}_{\lambda,\pi}) \in \chi^+_{T_q^{N_q}+4\mathbf{v}_{\lambda,\pi}}$ (or $X^-_l(T_q^{N_q} + 4\mathbf{v}_{\lambda,\pi}) \in \chi^-_{T_q^{N_q}+4\mathbf{v}_{\lambda,\pi}}$) for some $l \le q$, then $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(\lfloor \mathbf{n}_\lambda X_l \rfloor + i^{l,+}_{\mathbf{a}_\lambda(s-T_l)}) = 2$ (or $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(\lfloor \mathbf{n}_\lambda X_l \rfloor + i^{l,-}_{\mathbf{a}_\lambda(s-T_l)}) = 2$) for all $s \in [T_q^{N_q} + 4\mathbf{v}_{\lambda,\pi}\,, T_{q+1}]$ (because $T^{D,+}_l > T_{q+1} + 3\alpha$).

We can prove similar lemmas as Lemmas II.8.11 and II.8.12 replacing T_q^k by $T_q^{N_q}$ and T_q^{k+1} by T_{q+1}. Thus, Lemma II.8.15 follows exactly as in Step 4 and Step 5 in the proof of Corollary II.8.14. □

The proof of Lemma II.8.4 is completed.

II.8.5. Proof of Theorem II.6.1 for $p > 0$

We finally give the proof of the Theorem II.6.1 in the case $p > 0$.

Proof. Let us fix $x_0 \in (-A\,, A)$, $t_0 \in (0\,, T)$ and $\varepsilon > 0$. We will prove that with our coupling (see Subsection II.8.4.1), when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$, there holds that

(a) $\lim_{\lambda,\pi} \mathbb{P}\left[\boldsymbol{\delta}(D_{t_0}^{\lambda,\pi}(x_0), D_{t_0}(x_0)) > \varepsilon\right] = 0$;

(b) $\lim_{\lambda,\pi} \mathbb{P}\left[\boldsymbol{\delta}_T(D^{\lambda,\pi}(x_0), D(x_0)) > \varepsilon\right] = 0$;

(c) $\lim_{\lambda,\pi} \mathbb{P}\left[|Z_t^{\lambda,\pi}(x_0) - Z_t(x_0)| > \varepsilon\right] = 0$;

(d) $\lim_{\lambda,\pi} \mathbb{P}\left[\int_0^T |Z_t^{\lambda,\pi}(x_0) - Z_t(x_0)| \,\mathrm{d}t > \varepsilon\right] = 0$;

(e) $\lim_{\lambda,\pi} \mathbb{P}\left[|W_{t_0}^{\lambda,\pi}(x_0) - Z_{t_0}(x_0)| > \varepsilon\right] = 0$, where

$$W_{t_0}^{\lambda,\pi}(x_0) = \left(\frac{\log(|C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}, \lfloor \mathbf{n}_\lambda x_0 \rfloor)|)}{\log(1/\lambda)} \mathbf{1}_{\{|C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}, \lfloor \mathbf{n}_\lambda x_0 \rfloor)| \geq 1\}}\right) \wedge 1.$$

These points will clearly imply the result.

First, we introduce the event $\Omega_{A,T}^{x_0,t_0}(\alpha,\lambda,\pi)$ on which

(i) $x_0 \notin \cup_{y \in \mathcal{B}_M^D \cup \chi_{t_0}} (y - 3\alpha/p\,, y + 3\alpha/p)$;

(ii) for all $s \in \{T_k(x_0) : k = 1, \dots, n\} \cup \mathcal{T}_M \cup \mathcal{S}_M \cup \mathcal{S}_M^1 \cup \mathcal{S}_M^2$ with $s \leq t_0$, there holds that $t_0 - s > 3\alpha$;

(iii) if $t_0 \neq 1$, for all $s \in \{T_k(x_0) : k = 1, \dots, n\} \cup \mathcal{T}_M \cup \mathcal{S}_M \cup \mathcal{S}_M^1 \cup \mathcal{S}_M^2$ with $s \leq t_0$, there holds that $|t_0 - (s+1)| > 3\alpha$;

(iv) if $t_0 \geq 1$, for all $i \in I_A^\lambda$, $N^{S,\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) - N^{S,\lambda,\pi}_{\mathbf{a}_\lambda (t_0 - 1)}(i) > 0$;

(v) if $t_c = t_0 - \tau_{t_0-}(x_0) < 1$, there are

$$-\lfloor \lambda^{-(t_c+\alpha)} \rfloor < i_1 < -\lfloor \lambda^{-(t_c-\alpha)} \rfloor < 0 < \lfloor \lambda^{-(t_c-\alpha)} \rfloor < i_2 < \lfloor \lambda^{-(t_c+\alpha)} \rfloor$$

such that

- $N^{S,0}_{\mathbf{a}_\lambda t_0}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1) - N^{S,0}_{\mathbf{a}_\lambda(\tau_{t_0-}(x_0) - \mathbf{v}_{\lambda,\pi})}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1) = 0$ and $N^{S,0}_{\mathbf{a}_\lambda t_0}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2) - N^{S,0}_{\mathbf{a}_\lambda(\tau_{t_0-}(x_0) - \mathbf{v}_{\lambda,\pi})}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2) = 0$;
- for all $j \in [\![-\lfloor \lambda^{-(t_c-\alpha)} \rfloor, \lfloor \lambda^{-(t_c-\alpha)} \rfloor]\!]$, there holds that $N^{S,0}_{\mathbf{a}_\lambda t_0}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + j) - N^{S,0}_{\mathbf{a}_\lambda(\tau_{t_0-}(x_0) + \mathbf{v}_{\lambda,\pi})}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + j) > 0$.

Since $t_0 - \tau_{t_0-}(x_0) = 1$ occurs with positive probability only if $t_0 = 1$ (and $\tau_{t_0}(x_0) = 0$) the probability of the three first points clearly tend to 1 when α tends to 0. Since $(\tau_t(x_0))_{t \geq 0}$ is independent of $(N_t^{S,\lambda,\pi}(i))_{t \geq 0, i \in \mathbb{Z}}$ and since $(\tau_t(x_0))_{t \geq 0} \subset \{T_k(x_0) : k = 1, \dots, n\}$, the probability of the two last points tend to 1 as $\alpha \to 0$ and $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$, thanks to Lemma II.8.1-4,6,7. All this implies that for all $\delta > 0$, there is $\alpha > 0$ such that $\mathbb{P}\left[\Omega_{A,T}^{x_0,t_0}(\alpha,\lambda,\pi)\right] > 1 - \delta$ for all (λ,π) sufficiently close to the regime $\mathcal{R}(p)$.

Let us now fix $\delta > 0$. We consider $\alpha_0 \in (0\,,\varepsilon)$, $\gamma_0 \in (0\,,\alpha_0)$, $\lambda_0 \in (0\,,1)$ and $\epsilon_0 \in (0\,,1)$ such that for all $\lambda \in (0\,,\lambda_0)$ and all $\pi \geq 1$ in such a way that $|\mathbf{n}_\lambda/(\mathbf{a}_\lambda\pi) - p| < \epsilon_0$, we have

$$\mathbb{P}\left[\Omega(\alpha_0,\gamma_0,\lambda,\pi)\cap\Omega^{x_0,t_0}_{A,T}(\alpha_0,\lambda,\pi)\right] > 1-\delta.$$

We then consider $\lambda_1 \in (0\,,\lambda_0)$ and $\epsilon_1 \in (0\,,\epsilon_0)$ such that for all all $\lambda \in (0\,,\lambda_1)$ and all $\pi \geq 1$ in such a way that $|\mathbf{n}_\lambda/(\mathbf{a}_\lambda\pi) - p| < \epsilon_1$, we have

- $4(\mathbf{v}_{\lambda,\pi} + p(\mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi})/\mathbf{n}_\lambda) \leq \alpha_0$;
- $\alpha_0 + \log(\mathbf{a}_\lambda)/\log(1/\lambda) < \varepsilon$;
- $4(\mathbf{m}_\lambda + \mathbf{k}_{\lambda,\pi})/\mathbf{n}_\lambda < \varepsilon$;
- $1/(2\mathbf{m}_\lambda\lambda^{t_c-\varepsilon}) < \delta$ and $1/(2\mathbf{m}_\lambda\lambda^{t_c+\mathbf{v}_{\lambda,\pi}}) < \delta$ if $t_c < 1$.

All this can be done properly by using the fact that $\mathbf{v}_{\lambda,\pi} \to 0$ and $(\mathbf{m}_\lambda + \mathbf{k}_{\lambda,\pi})/\mathbf{n}_\lambda \to 0$.

In the rest of the proof, we consider $\lambda \in (0\,,\lambda_1)$ and $\pi \geq 1$ in such a way that $|\mathbf{n}_\lambda/(\mathbf{a}_\lambda\pi) - p| \leq \epsilon_1$. Observe that, on $\Omega(\alpha_0,\gamma_0,\lambda,\pi)$, there holds that $\tau_{t_0-}(x_0) = \tau_{t_0}(x_0)$ and $[x_0]_{\lambda,\pi} \cap \left(\bigcup_{x\in\mathcal{B}^D_M\cup\chi_{t_0}}[x]_{\lambda,\pi}\right) = \emptyset$.

Step 1. We first show that (a) (which holds for an arbitrary value of $t_0 \in (0\,,T)$) implies (b). Indeed, we have by construction, for any $t \in [0\,,T]$, $\boldsymbol{\delta}(D^{\lambda,\pi}_t(x_0), D_t(x_0)) < 4A$. Hence, by dominated convergence, (a) implies that $\lim_{\lambda,\pi}\mathbb{E}\left[\boldsymbol{\delta}(D^{\lambda,\pi}_t(x_0), D_t(x_0))\right] = 0$, whence again by dominated convergence, $\lim_{\lambda,\pi}\mathbb{E}\left[\boldsymbol{\delta}_T(D^{\lambda,\pi}(x_0), D(x_0))\right] = 0$.

Step 2. Next, (c) implies (d), exactly as in Step 1.

Step 3. Due to Lemma II.8.5, we know that, on $\Omega(\alpha_0,\gamma_0,\lambda,\pi)\cap\Omega^{x_0,t_0}_{A,T}(\alpha_0,\lambda,\pi)$, since $t_0 > \tau_{t_0}(x_0) + 3\alpha_0$, for all $i \in (x_0)_\lambda$,

$$\left|\rho^{\lambda,\pi}_{t_0}(i) - \tau_{t_0}(x_0)\right| \leq \mathbf{v}_{\lambda,\pi}.$$

For all $i \in (x_0)_\lambda$, since $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) \leq 1$, there holds

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) = \min(N^{S,\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) - N^{S,\lambda,\pi}_{\mathbf{a}_\lambda\rho^{\lambda,\pi}_{t_0}(i)}(i), 1).$$

Thus, for all $i \in (x_0)_\lambda$,

$$\underline{\eta}^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) \leq \eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) \leq \overline{\eta}^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i)$$

where

$$\underline{\eta}^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) := \min(N^{S,0}_{\mathbf{a}_\lambda t_0}(i) - N^{S,0}_{\mathbf{a}_\lambda(\tau_{t_0}(x_0)+\mathbf{v}_{\lambda,\pi})}(i), 1),$$
$$\overline{\eta}^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) := \min(N^{S,0}_{\mathbf{a}_\lambda t_0}(i) - N^{S,0}_{\mathbf{a}_\lambda(\tau_{t_0}(x_0)-\mathbf{v}_{\lambda,\pi})\vee 0}(i), 1).$$

We also recall that by construction, $(\tau_t(x_0))_{t\geq 0}$ is independent of $(N^{S,0}_t(i))_{t\geq 0, i\in\mathbb{Z}}$.

Step 4. Here we prove (e). We work on $\Omega(\alpha_0,\gamma_0,\lambda,\pi)\cap\Omega^{x_0,t_0}_{A,T}(\alpha_0,\lambda,\pi)$. By Step 3 and point (v) of the event $\Omega^{x_0,t_0}_{A,T}(\alpha_0,\lambda,\pi)$, we observe that if $0<t_c=t_0-\tau_{t_0}(x_0)<1$, then

$$\begin{aligned}\llbracket \lfloor \mathbf{n}_\lambda x_0\rfloor - \lfloor \lambda^{-(t_c-\alpha_0)}\rfloor \,,\, \lfloor \mathbf{n}_\lambda x_0\rfloor + \lfloor \lambda^{-(t_c-\alpha_0)}\rfloor \rrbracket &\subset C(\underline{\eta}^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}, \lfloor \mathbf{n}_\lambda x_0\rfloor)\\ &\subset C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}, \lfloor \mathbf{n}_\lambda x_0\rfloor) \subset C(\overline{\eta}^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}, \lfloor \mathbf{n}_\lambda x_0\rfloor)\subset \llbracket \lfloor \mathbf{n}_\lambda x_0\rfloor + i_1 \,,\, \lfloor \mathbf{n}_\lambda x_0\rfloor + i_2\rrbracket\\ &\subset \llbracket \lfloor \mathbf{n}_\lambda x_0\rfloor - \lfloor \lambda^{-(t_c+\alpha_0)}\rfloor \,,\, \lfloor \mathbf{n}_\lambda x_0\rfloor + \lfloor \lambda^{-(t_c+\alpha_0)}\rfloor \rrbracket.\end{aligned}$$

Thus, this implies that,

$$|W^{\lambda,\pi}_{t_0}(x_0)-(t_0-\tau_{t_0}(x_0))|\le \alpha_0+\frac{\log(2)}{\log(1/\lambda)}<\varepsilon.$$

If now $t_0-\tau_{t_0}(x_0)>1$, then $t_0-\tau_{t_0}(x_0)>1+3\alpha_0$ thanks to $\Omega^{x_0,t_0}_{A,T}(\alpha_0,\lambda,\pi)$. Then Step 3 and point (iv) of $\Omega^{x_0,t_0}_{A,T}(\alpha_0,\lambda,\pi)$ imply that $(x_0)_\lambda\subset C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}, \lfloor \mathbf{n}_\lambda x_0\rfloor)$ whence $|C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}, \lfloor \mathbf{n}_\lambda x_0\rfloor)|\ge 2\mathbf{m}_\lambda$. Consequently,

$$W^{\lambda,\pi}_{t_0}(x_0)\ge 1-\frac{\log(\mathbf{a}_\lambda)}{\log(1/\lambda)}>1-\varepsilon.$$

It only remains to study what happens when $t_0=1$. By construction, we have $\tau_{t_0}(x_0)=0$ and by Lemma II.8.5, we have $\rho^{\lambda,\pi}_{t_0}(i)=0$ for all $i\in(x_0)_\lambda$. By Step 3 and point (iv) of the event $\Omega^{x_0,t_0}_{A,T}(\alpha_0,\lambda,\pi)$, we deduce as above that $(x_0)_\lambda\subset C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}, \lfloor \mathbf{n}_\lambda x_0\rfloor)$ and conclude $|C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}, \lfloor \mathbf{n}_\lambda x_0\rfloor)|\ge 2\mathbf{m}_\lambda$ whence

$$W^{\lambda,\pi}_{t_0}(x_0)\ge 1-\frac{\log(\mathbf{a}_\lambda)}{\log(1/\lambda)}\ge 1-\varepsilon.$$

Recalling that $Z_{t_0}(x_0)=(t_0-\tau_{t_0}(x_0))\wedge 1$, we have proved that

$$\mathbb{P}\left[|W^{\lambda,\pi}_{t_0}(x_0)-Z_{t_0}(x_0))|<\varepsilon\right]\ge \mathbb{P}\left[\Omega(\alpha_0,\gamma_0,\lambda,\pi)\cap\Omega^{x_0,t_0}_{A,T}(\alpha_0,\lambda,\pi)\right]\ge 1-\delta,$$

as desired.

Step 5. Here we prove (c). Recall that $Z^{\lambda,\pi}_{t_0}(x_0)=\left(-\frac{\log(1-K^{\lambda,\pi}_{t_0}(x_0))}{\log(1/\lambda)}\right)\wedge 1$ where $K^{\lambda,\pi}_{t_0}(x_0)=(2\mathbf{m}_\lambda+1)^{-1}\left|\left\{i\in\llbracket\lfloor \mathbf{n}_\lambda X_0\rfloor-\mathbf{m}_\lambda\,,\,\lfloor \mathbf{n}_\lambda X_0\rfloor+\mathbf{m}_\lambda\rrbracket : \eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i)=1\right\}\right|$. We work on $\Omega(\alpha_0,\gamma_0,\lambda,\pi)\cap\Omega^{x_0,t_0}_{A,T}(\alpha_0,\lambda,\pi)$ and set $t_c=t_0-\tau_{t_0}(x_0)$.

Case 1. If $t_c\ge 1$, we have checked in Step 4 that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i)=1$ for all $i\in(x_0)_\lambda$, whence $K^{\lambda,\pi}_{t_0}(x_0)=1$ and $Z^{\lambda,\pi}_{t_0}(x_0)=1$.

Case 2. If now $0<t_c<1$, we deduce from Step 3 that

$$\underline{K}^{\lambda,\pi}_{t_0}(x_0)\le K^{\lambda,\pi}_{t_0}(x_0)\le \overline{K}^{\lambda,\pi}_{t_0}(x_0)$$

where

$$\underline{K}^{\lambda,\pi}_{t_0}(x_0) = (2\mathbf{m}_\lambda + 1)^{-1} \left|\left\{ i \in [\![\lfloor \mathbf{n}_\lambda X_0 \rfloor - \mathbf{m}_\lambda \,,\, \lfloor \mathbf{n}_\lambda x_0 \rfloor + \mathbf{m}_\lambda]\!] : \underline{\eta}^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) = 1 \right\}\right|,$$
$$\overline{K}^{\lambda,\pi}_{t_0}(x_0) = (2\mathbf{m}_\lambda + 1)^{-1} \left|\left\{ i \in [\![\lfloor \mathbf{n}_\lambda X_0 \rfloor - \mathbf{m}_\lambda \,,\, \lfloor \mathbf{n}_\lambda x_0 \rfloor + \mathbf{m}_\lambda]\!] : \overline{\eta}^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) = 1 \right\}\right|.$$

The random variable $\underline{X}^{\lambda,\pi}_{t_0}(x_0) = (2\mathbf{m}_\lambda + 1)\underline{K}^{\lambda,\pi}_{t_0}(x_0)$ has a binomial distribution with parameters $2\mathbf{m}_\lambda + 1$ and $1 - \lambda^{t_c - \mathbf{v}_{\lambda,\pi}}$. Then, using Bienaymé-Chebyshev's inequality,

$$\begin{aligned}
&\mathbb{P}\left[\underline{K}^{\lambda,\pi}_{t_0}(x_0) \le 1 - \lambda^{t_c - \varepsilon}\right] = \mathbb{P}\left[\underline{X}^{\lambda,\pi}_{t_0}(x_0) \le (2\mathbf{m}_\lambda + 1)(1 - \lambda^{t_c - \varepsilon})\right] \\
&\le \mathbb{P}\left[\left|\underline{X}^{\lambda,\pi}_{t_0}(x_0) - (2\mathbf{m}_\lambda + 1)(1 - \lambda^{t_c - \mathbf{v}_{\lambda,\pi}})\right| \ge (2\mathbf{m}_\lambda + 1)\left(\lambda^{t_c - \varepsilon} - \lambda^{t_c - \mathbf{v}_{\lambda,\pi}}\right)\right] \\
&\le \frac{(2\mathbf{m}_\lambda + 1)\left(1 - \lambda^{t_c - \mathbf{v}_{\lambda,\pi}}\right)\lambda^{t_c - \mathbf{v}_{\lambda,\pi}}}{(2\mathbf{m}_\lambda + 1)^2(\lambda^{t_c - \varepsilon} - \lambda^{t_c - \mathbf{v}_{\lambda,\pi}})^2} \\
&= \frac{1 - \lambda^{t_c - \mathbf{v}_{\lambda,\pi}}}{(2\mathbf{m}_\lambda + 1)\lambda^{t_c - \mathbf{v}_{\lambda,\pi}}(\lambda^{\mathbf{v}_{\lambda,\pi} - \varepsilon} - 1)^2} \simeq \frac{1}{2\mathbf{m}_\lambda \lambda^{t_c - 2\varepsilon + \mathbf{v}_{\lambda,\pi}}} \\
&\le \frac{1}{2\mathbf{m}_\lambda \lambda^{t_c - \varepsilon}} \text{ (because } 0 < \mathbf{v}_{\lambda,\pi} < \alpha_0 < \varepsilon) \\
&\le \delta.
\end{aligned}$$

By the same way, since $\overline{X}^{\lambda,\pi}_{t_0}(x_0) = (2\mathbf{m}_\lambda + 1)\underline{K}^{\lambda,\pi}_{t_0}(x_0)$ has a binomial distribution with parameters $2\mathbf{m}_\lambda + 1$ and $1 - \lambda^{t_c + \mathbf{v}_{\lambda,\pi}}$,

$$\begin{aligned}
&\mathbb{P}\left[\overline{K}^{\lambda,\pi}_{t_0}(x_0) \ge 1 - \lambda^{t_c + \varepsilon}\right] = \mathbb{P}\left[\overline{X}^{\lambda,\pi}_{t_0}(x_0) \ge (2\mathbf{m}_\lambda + 1)(1 - \lambda^{t_c + \varepsilon})\right] \\
&\le \mathbb{P}\left[\left|\overline{X}^{\lambda,\pi}_{t_0}(x_0) - (2\mathbf{m}_\lambda + 1)(1 - \lambda^{t_c + \mathbf{v}_{\lambda,\pi}})\right| \ge (2\mathbf{m}_\lambda + 1)\left(\lambda^{t_c + \mathbf{v}_{\lambda,\pi}} - \lambda^{t_c + \varepsilon}\right)\right] \\
&\le \frac{(2\mathbf{m}_\lambda + 1)\left(1 - \lambda^{t_c + \mathbf{v}_{\lambda,\pi}}\right)\lambda^{t_c + \mathbf{v}_{\lambda,\pi}}}{(2\mathbf{m}_\lambda + 1)^2(\lambda^{t_c + \mathbf{v}_{\lambda,\pi}} - \lambda^{t_c + \varepsilon})^2} \simeq \frac{1}{2\mathbf{m}_\lambda \lambda^{t_c + \mathbf{v}_{\lambda,\pi}}} \le \delta.
\end{aligned}$$

All this implies that,

$$\mathbb{P}\left[K^{\lambda,\pi}_{t_0}(x_0) \in (1 - \lambda^{t_c - \varepsilon} \,,\, 1 - \lambda^{t_c + \varepsilon})\right] \ge 1 - c\delta,$$

for some constante $c > 0$, whence

$$\mathbb{P}\left[Z^{\lambda,\pi}_{t_0}(x_0) \in (t_c - \varepsilon \,,\, t_c + \varepsilon)\right] \ge 1 - c\delta.$$

This is nothing but the goal, since $Z_{t_0}(x_0) = t_0 - \tau_{t_0}(x_0) = t_c$ as soon as $Z_{t_0}(x_0) < 1$.

Step 6. It remains to prove (a). On $\Omega(\alpha_0, \gamma_0, \lambda, \pi) \cap \Omega^{x_0,t_0}_{A,T}(\alpha_0, \lambda, \pi)$, we check that

(i) If $Z_{t_0}(x_0) < 1$, then $D_{t_0}(x_0) = \{x_0\}$ and $C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}, \lfloor \mathbf{n}_\lambda x_0 \rfloor) \subset (x_0)_\lambda$ (see Step 4 above), whence

$$D^{\lambda,\pi}_{t_0}(x_0) \subset [x_0 - \mathbf{m}_\lambda/\mathbf{n}_\lambda \,,\, x_0 + \mathbf{m}_\lambda/\mathbf{n}_\lambda].$$

We deduce that

$$\boldsymbol{\delta}(D^{\lambda,\pi}_{t_0}(x_0), D_{t_0}(x_0)) \le 2\mathbf{m}_\lambda/\mathbf{n}_\lambda.$$

(ii) If $Z_{t_0}(x_0) = 1$ and $D_{t_0}(x_0) = [a\,,b]$, for some $a, b \in \chi_{t_0}$, then

- for all $i \in [\![\lfloor \mathbf{n}_\lambda a\rfloor + \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}\,,\lfloor \mathbf{n}_\lambda b\rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}]\!] \setminus \left(\cup_{x\in\mathcal{B}_M^D}[x]_{\lambda,\pi}\right)$, $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) = 1$. Indeed, there is no burning tree in $[\![\lfloor \mathbf{n}_\lambda a\rfloor + \mathbf{k}_{\lambda,\pi}\,,\lfloor \mathbf{n}_\lambda b\rfloor - \mathbf{k}_{\lambda,\pi}]\!]$ at time $\mathbf{a}_\lambda t_0$ (use a very similar result as in Lemma II.8.6). Next, by construction, $Z_{t_0}(y) = 1$ for all $y \in (a\,,b)$ whence $\tau_{t_0}(y) \le t_0 - 1$. Using $\Omega^{x_0,t_0}_{A,T}(\alpha_0, \lambda, \pi)$, we deduce that $\tau_{t_0}(y) \le t_0 - 1 - 3\alpha_0$. Using finally Lemma II.8.5 and $\Omega^S_3(\lambda, \pi)$, we deduce the claim;
- for all $x \in \mathcal{B}^D_M \cap (a\,,b)$, and all $i \in [x]_{\lambda,\pi}$, $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) = 1$. Indeed, on $\Omega^{x_0,t_0}_{A,T}(\alpha_0, \lambda, \pi)$, we have $\tilde{H}_{t_0-}(x) = 0$ whence $\tau_{t_0}(x_0) \le t_0 - 1 - 3\alpha_0$. We deduce that no match falling outside $[x]_{\lambda,\pi}$ affect this zone during the time interval $[\mathbf{a}_\lambda(t_0 - 1 - \alpha_0)\,, \mathbf{a}_\lambda t_0]$ and conclude by distinguishing several cases, as in Step 3 in the proof of Lemma II.8.12;
- if $a \in \chi^+_{t_0} \cup \chi^-_{t_0}$ there is $i \in \langle a\rangle_{\lambda,\pi}$ such that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) = 2$ (thanks to $\Omega^{\lambda,\pi}_T$, since on $\Omega^{x_0,t_0}_{A,T}(\alpha_0, \lambda, \pi)$, we have $|t_0 - s| \ge 3\alpha$ for all $s \in \mathcal{T}^D_M$) whereas if $a \in \chi^0_{t_0}$, there is $i \in (a)_\lambda$ such that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) = 0$ (use similar argument as in Lemma II.8.13, observing that $|t_0 - s| \ge 3\alpha$ for all $s \in \mathcal{T}^D_M$). Similar observation of course holds for b;

so that

$$[\![\lfloor \mathbf{n}_\lambda a\rfloor + \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}\,,\lfloor \mathbf{n}_\lambda b\rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}]\!] \subset C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}, \lfloor \mathbf{n}_\lambda x_0\rfloor)$$
$$\subset [\![\lfloor \mathbf{n}_\lambda a\rfloor - \mathbf{m}_\lambda - \mathbf{k}_{\lambda,\pi}\,,\lfloor \mathbf{n}_\lambda b\rfloor + \mathbf{m}_\lambda + \mathbf{k}_{\lambda,\pi}]\!]$$

and thus

$$[a + \frac{\mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}}{\mathbf{n}_\lambda}\,, b - \frac{\mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}}{\mathbf{n}_\lambda}] \subset D^{\lambda,\pi}_{t_0}(x_0) \subset [a - \frac{\mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}}{\mathbf{n}_\lambda}\,, b + \frac{\mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}}{\mathbf{n}_\lambda}],$$

whence $\boldsymbol{\delta}(D^{\lambda,\pi}_{t_0}(x_0), D_{t_0}(x_0)) \le 4(\mathbf{m}_\lambda + \mathbf{k}_{\lambda,\pi})/\mathbf{n}_\lambda$.

Thus, on $\Omega(\alpha_0, \gamma_0, \lambda, \pi) \cap \Omega^{x_0,t_0}_{A,T}(\alpha_0, \lambda, \pi)$, we always have $\boldsymbol{\delta}(D^{\lambda,\pi}_{t_0}(x_0), D_{t_0}(x_0)) \le 4(\mathbf{m}_\lambda + \mathbf{k}_{\lambda,\pi})/\mathbf{n}_\lambda$. We conclude that

$$\mathbb{P}\left[\boldsymbol{\delta}(D^{\lambda,\pi}_{t_0}(x_0), D_{t_0}(x_0)) \le \varepsilon\right] \ge \mathbb{P}\left[\Omega(\alpha_0, \gamma_0, \lambda, \pi) \cap \Omega^{x_0,t_0}_{A,T}(\alpha_0, \lambda, \pi)\right] \ge 1 - \delta.$$

This concludes the proof of Theorem II.6.1 for $p > 0$. $\square$

II.8.6. Cluster size distribution when $p > 0$

The aim of this section is to prove Corollary II.2.6 when $p > 0$.

II.8.6.1. Study of the LFFP(p)

Recall Subsection II.1.2 and Definition II.2.1.

Definition II.8.16. *Let $(Z_t(x), H_t(x), F_t(x))_{t\geq 0, x\in\mathbb{R}}$ be a LFFP(p). For all $x \in \mathbb{R}$ and all $t \geq 0$, we define*

$$\mathscr{D}_t(x) = [\mathscr{L}_t(x)\,, \mathscr{R}_t(x)]$$

where

$$\mathscr{L}_t(x) = \inf\left\{y \leq x : \forall (r,v) \in \Lambda^p_{(x,t)}(y, t - p(x-y))\,,\ Z_{v-}(r) = 1 \text{ and } H_{v-}(r) = 0\right\},$$
$$\mathscr{R}_t(x) = \sup\left\{y \geq x : \forall (r,v) \in \Lambda^p_{(x,t)}(y, t + p(x-y))\,,\ Z_{v-}(r) = 1 \text{ and } H_{v-}(r) = 0\right\}.$$

Observe that for all $t \in [0\,, T]$ and all $x \in \mathbb{R}$,

- $Z_t(x) = 0$ if and only if $\pi_M\left((\mathscr{D}_{t-}(x) \times \mathbb{R}) \cap \Lambda^p_{(x,t)}\right) > 0$;
- $\mathscr{D}_t(x) = \{x\}$ if $t \in [0\,, 1)$;
- $|\mathscr{D}_t(x)| \leq 2(t-1)/p$.

Lemma II.8.17. *Let $(Z_t(x), H_t(x), F_t(x))_{t\geq 0, x\in\mathbb{R}}$ be a LFFP(p) and consider $(D_t(x))_{t\geq 0, x\in\mathbb{R}}$ and $(\mathscr{D}_t(x))_{t\geq 0, x\in\mathbb{R}}$ the associated processes. There are some constants $0 < c_1 < c_2$ and $0 < \kappa_1 < \kappa_2$, depending only on p, such that the following estimates hold.*

(i) For any $t \in (1, \infty)$, any $x \in \mathbb{R}$, any $z \in [0, 1)$, $\mathbb{P}[Z_t(x) = z] = 0$.

(ii) For any $t \in [0, \infty)$, any $B > 0$, any $x \in \mathbb{R}$, $\mathbb{P}[|D_t(x)| = B] = 0$.

(iii) For all $t \in [0, \infty)$, all $x \in \mathbb{R}$, all $B > 0$, $\mathbb{P}[|D_t(x)| \geq B] \leq c_2 e^{-\kappa_1 B}$.

(iv) For all $t \in [\frac{11}{8}, \infty)$, all $x \in \mathbb{R}$, all $B > 0$, $\mathbb{P}[|D_t(x)| \geq B] \geq c_1 e^{-\kappa_2 B}$.

(v) For all $t \in [0, \infty)$, all $x \in \mathbb{R}$, all $B > 0$, $\mathbb{P}[|\mathscr{D}_t(x)| \geq B] \leq c_2 e^{-\kappa_1 B}$.

(vi) For all $t \in [\frac{3}{2}, \infty)$, all $x \in \mathbb{R}$, all $B \in (0\,, (2t-3)/p)$, $\mathbb{P}[|\mathscr{D}_t(x)| \geq B] \geq c_1 e^{-\kappa_2(B+B^2)}$.

(vii) For all $t \in [(5+p)/2, \infty)$, all $0 \leq a < b < 1$, all $x \in \mathbb{R}$,

$$c_1(b-a) \leq \mathbb{P}[Z_t(x) \in [a\,, b]] \leq c_2(b-a).$$

Proof. By invariance by translation, it suffices to treat the case $x = 0$.

Point (i). For $t \in [0, 1]$, we have a.s. $Z_t(0) = t$. But for $t > 1$ and $z \in [0, 1)$, $Z_t(0) = z$ implies that a fire has crossed 0 at time $t - z$, so that necessarily $\pi_M(\Lambda^p_{(0,t)}) > 0$, recall Subsection II.1.2. This happens with probability 0.

Point (ii). For any $t > 0$, $|D_t(0)|$ is either 0 or of the form $|x - y|$, for some $x, y \in \chi_t$. We easily conclude as previously that for $B > 0$, $\Pr(|D_t(0)| = B) = 0$.

Point (iii). First if $t \in [0,1)$, we have a.s. $|D_t(0)| = 0$ and the result is obvious. Recall that for (X,τ) a mark of π_M, we have $H_t(X) > 0$ or $Z_t(X) < 1$ for all $t \in [\tau, \tau + 1/2)$ (see the proof of Proposition II.3.5-Step 1). This implies that for $t \geq 1$,

$$\begin{aligned}\{|D_t(0)| \geq B\} &\subset \{[0,B/2] \text{ is connected at time } t \text{ or } [-B/2,0] \text{ is connected at time } t\} \\ &\subset \{\pi_M([0,B/2] \times [t-1/4,t]) = 0\} \cup \{\pi_M([-B/2,0] \times [t-1/4,t]) = 0\}\,.\end{aligned}$$

Consequently, $\Pr[|D_t(0)| \geq B] \leq 2e^{-B/8}$ as desired.

Point (iv). Fix $B > 0$ and $t \geq 11/8$. Set $\Delta = \frac{3}{16p}$ and $K = \left\lfloor \frac{1}{\Delta}\left(B + \frac{11}{4p}\right)\right\rfloor + 1$. Consider the event $\Omega_{t,B} = \Omega^0_{t,B} \cap \bigcap_{k=0}^{K-1} \Omega_{t,B,k}$, illustrated by Figure II.8, where

- $\Omega^0_{t,B} = \{\pi_M([-5/(4p), B + 5/(4p)] \times [t-5/4\,, t]) = 0\}$;
- for all $k \in [\![0\,, K-1]\!]$, $\Omega_{t,B,k} = \{\pi_M(D_k) = 1\} \cap \{\pi_M\,(C_k \setminus D_k) = 0\}$ where

$$\begin{aligned} C_k &= \left[-\frac{11}{8p} + k\Delta, -\frac{11}{8p} + (k+1)\Delta\right] \times [t-11/8\,, t-5/4] \\ D_k &= \left[-\frac{11}{8p} + (k+\frac{1}{3})\Delta, -\frac{11}{8p} + (k+\frac{2}{3})\Delta\right] \times [t-11/8\,, t-5/4], \end{aligned}$$

  see Figure II.9. Observe that $\bigcup_{k=0}^{K-1} C_k \supset [-11/(8p)\,, B + 11/(8p)]$.

We have $\mathbb{P}\left[\Omega^0_{t,B}\right] = \exp\left(-\frac{5}{4}(B + \frac{5}{2p})\right)$ whence for all $k \in [\![0\,, K-1]\!]$, $\mathbb{P}\left[\Omega_{t,B,k}\right] = \frac{\Delta}{24} \times e^{-\frac{\Delta}{24}} \times e^{-\frac{\Delta}{12}}$. All these events being independent, we conclude that

$$\mathbb{P}\left[\Omega_{t,B}\right] = \exp\left(-\frac{5}{4}(B + \frac{5}{2p})\right) \times \left(\frac{\Delta}{24}e^{-\frac{\Delta}{8}}\right)^K \geq c_1 e^{-\kappa_2 B}$$

for some constant c_1 and κ_2 not depending on B. To conclude the proof of (iv), it thus suffices to check that $\Omega_{t,B} \subset \{[0,B] \subset D_t(0)\}$. But on $\Omega_{t,B}$, using the same arguments as in Point (iii), we observe that:

- for (X,τ) a mark of π_M, $H^A_s(X) > 0$ or $Z^A_s(X) < 1$ for all $s \in [\tau\,, \tau + 3/8]$. Thus, for all $k \in [\![0\,, K-1]\!]$, there is $x \in D_k$ such that $H^A_s(x) > 0$ or $Z^A_s(x) < 1$ for all $s \in [t-5/4\,, t-1]$;
- calling (X_k, τ_k) the mark of π_M in D_k, we have $\tau_k + p(X_{k+1} - X_k) \in [t-5/4\,, t-1]$ and $\tau_k + p(X_k - X_{k-1}) \in [t-5/4\,, t-1]$, see Figure II.9. Thus, if the fire starting on X_k at time τ_k is macroscopic, it is (at least) stopped by the marks (X_{k-1}, τ_{k-1}) and (X_{k+1}, τ_{k+1}) and does not affect the zone $[0\,, B]$ after $t-1$;
- for (Y,S) a mark of π_M such that $(Y,S) \notin [-11/(8p)\,, B + 11/(8p)] \times [t-11/8\,, t]$ and $Y + (t-S)/p \in [0\,, B]$, then there exists $k \in [\![0\,, K-1]\!]$ such that

$$Y + \frac{t - 11/8 - S}{p} \in \left[-\frac{11}{8p} + (k - \frac{1}{3})\Delta,\ -\frac{11}{8p} + (k + \frac{2}{3})\Delta\right].$$

We immediately conclude that $S + p(X_{k+1} - Y) \in [t - 5/4\,, t - 1]$. Thus, the right front of (Y, S) is stopped by the match (X_{k+1}, τ_{k+1}) and does not affect the zone $[0\,, B]$ after $t - 1$;

- for (Y, S) a mark of π_M such that $(Y, S) \notin [-11/(8p)\,, B + 11/(8p)] \times [t - 11/8\,, t]$ and $Y - (t - S)/p \in [0\,, B]$, we prove as above that the left front of (Y, S) is stopped by such a match (X_{k-1}, τ_{k-1}) and does not affect the zone $[0\,, B]$ after $t - 1$;
- by construction, the other fires may not affect the zone $[-11/(8p)\,, B + 11/(8p)]$ during the time interval $[t - 1\,, t]$.

As a conclusion, the zone $[0\,, B]$ is not affected by any fire during $[t - 1\,, t]$. Since the length of this time interval is greater than 1, we deduce that for all $x \in [0, B]$, $Z_t(x) = \min(Z_{t-1}(x) + 1, 1) = 1$ and $H_t(x) = \max(H_{t-1}(x) - 1, 0) = 0$, whence $[0, B] \subset D_t(0)$.

Point (v) First if $t \in [0, 1)$, we have a.s. $|\mathscr{D}_t(0)| = 0$ and the result is obvious. If $t \geq 1$ and $B > 2(t - 1)/p$,

$$\mathbb{P}\left[|\mathscr{D}_t(0)| \geq B\right] = 0.$$

Recall that for (X, τ) a mark of π_M, we have $H_t(X) > 0$ or $Z_t(X) < 1$ for all $t \in [\tau, \tau + 1/2)$ (see the proof of Proposition II.3.5-Step 1). This implies that for $t \geq 1$ and $B \in (0\,, 2(t - 1)/p)$,

$$\begin{aligned}\{|\mathscr{D}_t(0)| \geq B\} &\subset \{[0, B/2] \subset [0\,, \mathscr{R}_t(x)] \text{ or } [-B/2, 0] \subset [\mathscr{L}_t(x)\,, 0]\} \\ &\subset \left\{\pi_M\left(\left\{(r, v) \in \Lambda^p_{(0,s)}(B/2, s - pB/2) : s \in [t - 1/4\,, t]\right\}\right) = 0\right\} \\ &\qquad \cup \left\{\pi_M\left(\left\{(r, v) \in \Lambda^p_{(0,s)}(-B/2, s - pB/2) : s \in [t - 1/4\,, t]\right\}\right) = 0\right\}.\end{aligned}$$

Consequently, $\mathbb{P}\left[|\mathscr{D}_t(0)| \geq B\right] \leq 2e^{-B/8}$, as desired.

Point (vi) Let $t \geq 3/2$ and $B \in (0\,, (2t - 3)/p)$. From Point (iv), using space/time stationarity, we define an event $\tilde{\Omega}_{t,B}$, depending on the Poisson measure $\pi_M(\mathrm{d}x, \mathrm{d}s)$ restricted to $[-B/2 - 11/(8p)\,, B/2 + 11/(8p)] \times [t - pB/2 - 3/2\,, t - pB/2]$, on which $D_{t-pB/2}(0) \supset [-B/2\,, B/2]$. Next consider the event

$$\tilde{\Omega}^0_{t,B} = \{\pi_M\left([-B/2\,, B/2] \times [t - pB/2\,, t]\right) = 0\}.$$

We have $\mathbb{P}\left[\tilde{\Omega}^0_{t,B}\right] = e^{-pB^2/2}$.

The events $\tilde{\Omega}_{t,B}$ and $\tilde{\Omega}^0_{t,B}$ are independent, thus we have, recalling point (iv)

$$\mathbb{P}\left[\tilde{\Omega}_{t,B} \cap \tilde{\Omega}^0_{t,B}\right] = \mathbb{P}\left[\tilde{\Omega}_{t,B}\right] \times \mathbb{P}\left[\tilde{\Omega}^0_{t,B}\right] \geq c_1 e^{-\kappa_2(B+B^2)}.$$

Finally, we observe that for $(X, t - pB/2)$ a fire a time $t - pB/2$ with, for example, $X < -B/2$, we have, by construction, $X + (t - (t - pB/2))/p < 0$. Thus,

$$\tilde{\Omega}_{t,B} \cap \tilde{\Omega}^0_{t,B} \subset \{|\mathscr{D}_t(0)| \geq B\}.$$

This concludes the point.

Point (vii) For $0 \le a \le b < 1$ and $t \ge 1$, we have $Z_t(0) \in [a\,,b]$ if and only if there is $\tau \in [t-b\,,t-a]$ such that $Z_\tau(0) = 0$. And this happens if and only if

$$X_{t,a,b} := \int_{t-b}^{t-a} \int_{\mathbb{R}} \mathbf{1}_{\{(y,s-p|x-y|) \in \mathscr{D}_{s-}(0) \times [0,s]\}} \pi_M(\mathrm{d}y, \mathrm{d}s) \ge 1.$$

We deduce that

$$\mathbb{P}\left[Z_t(0) \in [a\,,b]\right] = \mathbb{P}\left[X_{t,a,b} \ge 1\right] \le \mathbb{E}\left[X_{t,a,b}\right] = \int_{t-b}^{t-a} \mathbb{E}\left[|\mathscr{D}_s(0)|\right] \mathrm{d}s \le C(b-a),$$

where we used Point (v) for the last inequality.

Next, we have $\{\pi_M(\mathscr{D}_{t-b}(0) \times [t-b\,,t-a]) \ge 1\} \subset \{X_{t,a,b} \ge 1\}$: it suffices to note that a.s.,

$$\begin{aligned} \{X_{t,a,b} = 0\} \subset \{X_{t,a,b} = 0, \mathscr{D}_{t-b}(0) \subset \mathscr{D}_s(0) \text{ for all } s \in [t-b\,,t-a]\} \\ \subset \{\pi_M(\mathscr{D}_{t-b}(0) \times [t-b\,,t-a]) = 0\}. \end{aligned}$$

Since now $\mathscr{D}_{t-b}(0)$ is independent of $\pi_M(\mathrm{d}x, \mathrm{d}s)$ restricted to $\mathbb{R} \times (t-b\,,\infty)$, we deduce that for $t \ge (5+p)/2$

$$\begin{aligned} \mathbb{P}\left[Z_t(0) \in [a\,,b]\right] &\ge \mathbb{P}\left[\pi_M(\mathscr{D}_{t-b}(0) \times [t-b\,,t-a]) \ge 1\right] \\ &\ge \mathbb{P}\left[|\mathscr{D}_{t-b}(0)| \ge 1\right](1-e^{-(b-a)}) \\ &\ge c(1-e^{-(b-a)}), \end{aligned}$$

where we used Point (vi) (here $t-b \ge 3/2$ and $(2t-3)/p \ge 1$) to get the last inequality. This concludes the proof, since $1-e^{-x} \ge x/2$ for all $x \in [0\,,1]$. ☐

II.8.6.2. Proof of Corollary II.2.6 when $p > 0$

We finally give the

Proof of Corollary II.2.6 when $p > 0$. For each $\lambda \in (0\,,1)$ and each $\pi \ge 1$, consider a (λ,π)−FFP $(\eta_t^{\lambda,\pi}(i))_{t\ge 0, i\in\mathbb{Z}}$. Let also $(Z_t(x), H_t(x), F_t(x))_{t\ge 0, x\in\mathbb{R}}$ be a LFFP(p) and consider the corresponding process $(D_t(x))_{t\ge 0, x\in\mathbb{R}}$.

Point (b). Using Lemma II.8.17-(iii)-(iv) and recalling that $|C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}, 0)|/\mathbf{n}_\lambda = |D_t^{\lambda,\pi}(0)|$, it suffices to check that for all $t \ge 3/2$ and all $B > 0$, when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$,

$$\lim_{\lambda,\pi} \mathbb{P}\left[|D_t^{\lambda,\pi}(0)| \ge B\right] = \mathbb{P}\left[|D_t(0)| \ge B\right].$$

This follows from Theorem II.2.4-2, which implies that $|D_t^{\lambda,\pi}(0)|$ goes in law to $|D_t(0)|$ and from Lemma II.8.17-(ii).

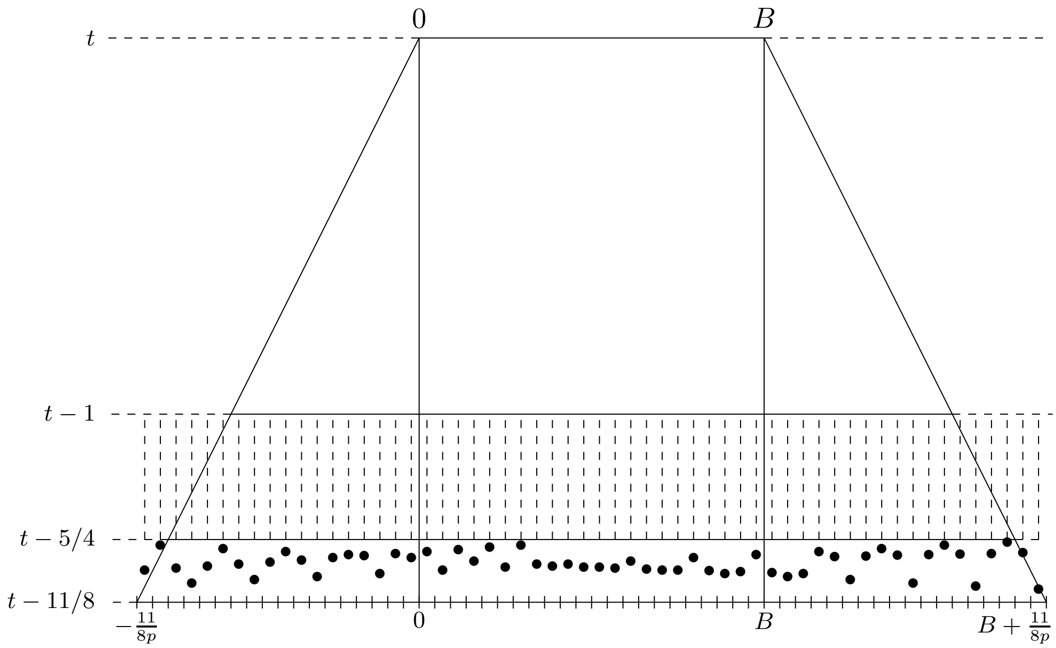

Figure II.8.: The event $\Omega_{t,B}$.

The marks of π_M are represented by $\bullet$. A match falls on each zone $D_k \subset C_k$.

Point (a). Due to Lemma II.8.17-(v) we only need that for all $0 < a < b < 1$, all $t \geq (5+p)/2$, when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$,

$$\lim_{\lambda,\pi} \mathbb{P}\left[|C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}, 0)| \in [\lambda^{-a}, \lambda^{-b}]\right] = \mathbb{P}\left[Z_t(0) \in [a, b]\right].$$

But using Theorem II.2.4-3 and Lemma II.8.17-(i), we know that

$$\lim_{\lambda,\pi} \mathbb{P}\left[\frac{\log(|C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}, 0)|)}{\log(1/\lambda)} \mathbf{1}_{\{|C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}, 0)| \geq 1\}} \in [a, b]\right] = \mathbb{P}\left[Z_t(0) \in [a, b]\right]$$

when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(p)$. One immediately concludes. $\square$

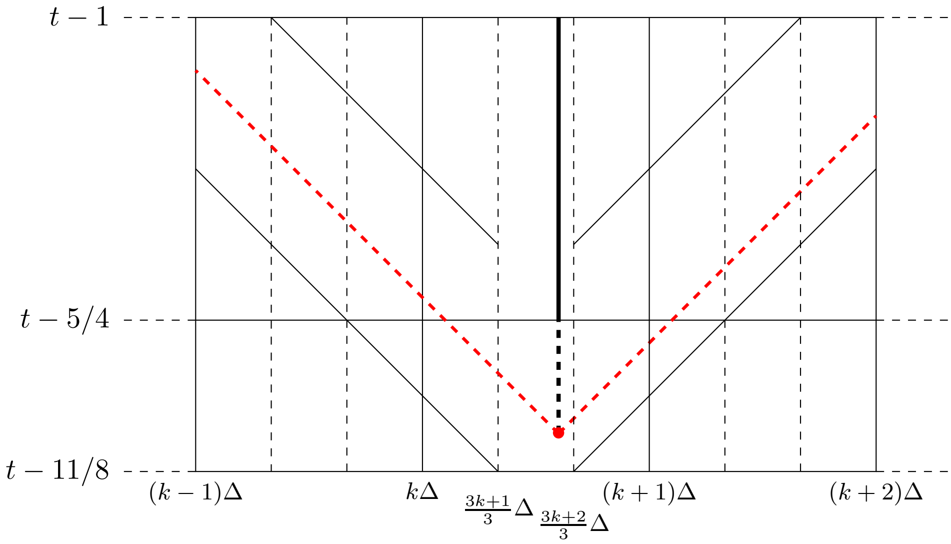

Figure II.9.: The event $\Omega_{t,B,k}$.

A match falls on $D_k = \left[-\frac{11}{8p} + (k+\frac{1}{3})\Delta, -\frac{11}{8p} + (k+\frac{2}{3})\Delta\right] \times [t-11/8\,, t-5/4]$ and is represented by $\bullet$. The dashed slope lines stand for the hypothetical fronts of the fire. The plain slope lines stand for the upper and lower possible positions of the fronts. The plain vertical thick line is the possible microscopic zone due to the fire in D_k. Thus, if the match falling on D_k is macroscopic, it is necessarily stopped by a microscopic zone in D_{k+1} and in D_{k-1}, since $H_s(X_{k+1}) > 0$ or $Z_s(X_{k+1}) < 1$ for all $s \in [t-5/4\,, t-1]$ and $H_s(X_{k-1}) > 0$ or $Z_s(X_{k-1}) < 1$ for all $s \in [t-5/4\,, t-1]$.

II.9. Convergence in the fast regime

The aim of this section is to prove Theorem II.6.1 when $p = 0$ and this will conclude the proof of Theorem II.2.4.

In the whole section, we fix the parameters $A > 0$ and $T > 2$. We omit the subscript/superscript A in the whole proof. The proof follows the ideas of the Section II.8.

We recall that $\mathbf{a}_\lambda = \log(1/\lambda)$, $\mathbf{n}_\lambda = \lfloor 1/(\lambda \mathbf{a}_\lambda) \rfloor$, $\mathbf{m}_\lambda = \lfloor 1/(\lambda \mathbf{a}_\lambda^2) \rfloor$, $\varepsilon_\lambda = 1/\mathbf{a}_\lambda^3$. We set as usual $A_\lambda = \lfloor \mathbf{n}_\lambda A \rfloor$ and $I_A^\lambda = [\![-A_\lambda \, , A_\lambda]\!]$. For $i \in \mathbb{Z}$, we set $i_\lambda = [i/\mathbf{n}_\lambda \, , (i+1)/\mathbf{n}_\lambda)$. For $[a \, , b]$ an interval of $[-A \, , A]$ and $\lambda \in (0 \, , 1)$, we recall, assuming that $-A < a < b < A$, that

$$
\begin{aligned}
[a \, , b]_\lambda &= [\![\lfloor \mathbf{n}_\lambda a \rfloor + \mathbf{m}_\lambda + 1 \, , \lfloor \mathbf{n}_\lambda b \rfloor - \mathbf{m}_\lambda - 1]\!] \subset \mathbb{Z}, \\
[-A \, , b]_\lambda &= [\![-A_\lambda \, , \lfloor \mathbf{n}_\lambda b \rfloor - \mathbf{m}_\lambda - 1]\!] \subset \mathbb{Z}, \\
[a \, , A]_\lambda &= [\![\lfloor \mathbf{n}_\lambda a \rfloor + \mathbf{m}_\lambda + 1 \, , A_\lambda]\!] \subset \mathbb{Z}.
\end{aligned}
$$

For $\lambda \in (0 \, , 1)$ and $\pi \geq 1$, we set

$$
\varkappa_{\lambda,\pi} = \frac{2 \mathbf{n}_\lambda A}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda .
$$

For $x \in (-A \, , A)$, $\lambda \in (0 \, , 1)$ and $\pi \geq 1$, we also recall that

$$
(x)_\lambda = [\![\lfloor \mathbf{n}_\lambda x \rfloor - \mathbf{m}_\lambda \, , \lfloor \mathbf{n}_\lambda x \rfloor + \mathbf{m}_\lambda]\!] \subset \mathbb{Z}.
$$

II.9.1. Occupation of vacant zone

For simplicity, we recall Lemma II.8.1.

Lemma II.9.1. *Consider a family of i.i.d. Poisson processes $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$. Let $a < b$.*

1. *For $t < 1$, $\lim_{\lambda \to 0} \mathbb{P}\left[\forall i \in [\![\lfloor a \mathbf{m}_\lambda \rfloor \, , \lfloor b \mathbf{m}_\lambda \rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) > 0 \right] = 0$;*
2. *For $t \geq 1$, $\lim_{\lambda \to 0} \mathbb{P}\left[\forall i \in [\![\lfloor a \mathbf{m}_\lambda \rfloor \, , \lfloor b \mathbf{m}_\lambda \rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) > 0 \right] = 1$;*
3. *For $t < 1$, $\lim_{\lambda \to 0} \mathbb{P}\left[\forall i \in [\![\lfloor a \mathbf{n}_\lambda \rfloor \, , \lfloor b \mathbf{n}_\lambda \rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) > 0 \right] = 0$;*
4. *For $t \geq 1$, $\lim_{\lambda \to 0} \mathbb{P}\left[\forall i \in [\![\lfloor a \mathbf{n}_\lambda \rfloor \, , \lfloor b \mathbf{n}_\lambda \rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) > 0 \right] = 1$;*
5. *For $t > 0$, $\lim_{\lambda \to 0} \mathbb{P}\left[\exists i \in [\![\lfloor a \mathbf{n}_\lambda \rfloor \, , \lfloor b \mathbf{n}_\lambda \rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) > 0 \right] = 1$;*
6. *For $t > 0$ and $\delta > 0$, $\lim_{\lambda \to 0} \mathbb{P}\left[\forall i \in [\![-\lfloor \lambda^{-(t+\delta)} \rfloor \, , \lfloor \lambda^{-(t+\delta)} \rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) > 0 \right] = 0$;*
7. *For $t > 0$ and $\delta > 0$, $\lim_{\lambda \to 0} \mathbb{P}\left[\forall i \in [\![-\lfloor \lambda^{-(t-\delta)} \rfloor \, , \lfloor \lambda^{-(t-\delta)} \rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) > 0 \right] = 1$.*

II.9.2. Height of the barrier

We describe here the time needed for a destroyed microscopic cluster to be regenerated. Roughly, we assume that the zone $(x_1)_\lambda$ around $\lfloor \mathbf{n}_\lambda x_1 \rfloor$, for some $x_1 \in [-A\,, A]$, has been made vacant at some time $\mathbf{a}_\lambda t_0$. Then we consider the situation where a match falls on $\lfloor \mathbf{n}_\lambda x_1 \rfloor$ at some time $\mathbf{a}_\lambda t_1 \in (\mathbf{a}_\lambda t_0\,, \mathbf{a}_\lambda(t_0+1))$ and we compute the delay needed for the destroyed cluster to be fully regenerated. As in Subsection II.8.2, we have to distinguish the cases $t_0 = 0$ and $t_0 > 1$.

Lemma II.9.2. *Consider two Poisson processes $(N_t^S(i))_{t\geq 0, i\in\mathbb{Z}}$ and $(N_t^P(i))_{t\geq 0, i\in\mathbb{Z}}$ with respective rates 1 and π, all these processes being independent. Consider also $\mathcal{M} := ((x_0,t_0),(x_1,t_1))$ with $x_0, x_1 \in (-A\,, A)$, $t_0 \in \{0\} \cup (1\,, \infty)$ and $t_1 \in (t_0\,, t_0+1)$. For $i \in I_A^\lambda$ and $t \geq 0$, we consider the process*

$$
\begin{aligned}
\zeta_t^{\lambda,\pi,\mathcal{M}}(i) = {} & \left(1 + \boldsymbol{1}_{\{t\geq \mathbf{a}_\lambda t_0, i = \lfloor \mathbf{n}_\lambda x_0 \rfloor\}}\right) \times \boldsymbol{1}_{\{t_0 > 1\}} \\
& + \boldsymbol{1}_{\{t\geq \mathbf{a}_\lambda t_1, i=\lfloor \mathbf{n}_\lambda x_1\rfloor, \zeta^{\lambda,\pi,\mathcal{M}}_{\mathbf{a}_\lambda t_1 -}(\lfloor \mathbf{n}_\lambda x_1 \rfloor)=1\}} + \int_0^t \boldsymbol{1}_{\{\zeta_{s-}^{\lambda,\pi,\mathcal{M}}(i)=0\}}\,\mathrm{d}N_s^S(i) \\
& + \int_0^t \boldsymbol{1}_{\{\zeta_{s-}^{\lambda,\pi,\mathcal{M}}(i+1)=2, \zeta_{s-}^{\lambda,\pi,\mathcal{M}}(i)=1\}}\,\mathrm{d}N_s^P(i+1) \\
& + \int_0^t \boldsymbol{1}_{\{\zeta_{s-}^{\lambda,\pi,\mathcal{M}}(i-1)=2, \zeta_{s-}^{\lambda,\pi,\mathcal{M}}(i)=1\}}\,\mathrm{d}N_s^P(i-1) \\
& - 2\int_0^t \boldsymbol{1}_{\{\zeta_{s-}^{\lambda,\pi,\mathcal{M}}(i)=2\}}\,\mathrm{d}N_s^P(i)
\end{aligned}
$$

with the convention $\zeta_t^{\lambda,\pi,\mathcal{M}}(\lfloor \mathbf{n}_\lambda A\rfloor + 1) = \zeta_t^{\lambda,\pi,\mathcal{M}}(-\lfloor \mathbf{n}_\lambda A \rfloor - 1) = 0$ for all $t \in [0\,,\infty)$.

Using the Poisson processes $(N^P(i))_{t\geq 0, i\in\mathbb{Z}}$, consider the burning times $(T_i^1)_{i\in\mathbb{Z}}$ of the propagation processes iginited at (x_1,t_1), recall Definition II.4.6, and define the destroyed cluster due to the match falling in $\lfloor \mathbf{n}_\lambda x_1 \rfloor$ at time $\mathbf{a}_\lambda t_1$, recall Definition II.4.8,

$$
C^P((\zeta_t^{\lambda,\pi}(i))_{t\geq 0, i\in\mathbb{Z}}, (x_1,t_1)) := [\![\lfloor \mathbf{n}_\lambda x_1\rfloor + i^g\,, \lfloor \mathbf{n}_\lambda x_1 \rfloor + i^d]\!].
$$

We finally define the time needed for $C^P((\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t\geq 0, i\in\mathbb{Z}}, (x_1,t_1))$ to become again occupied

$$
\Theta_{\mathcal{M}}^{\lambda,\pi} := \inf\left\{t > t_1 : \forall i \in C^P((\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t\geq 0, i\in\mathbb{Z}}, (x_1,t_1)), \zeta_{\mathbf{a}_\lambda t}^{\lambda,\pi,\mathcal{M}}(i) = 1\right\}.
$$

For all $\delta > 0$, there holds that,

$$
\lim_{\lambda,\pi} \mathbb{P}\left[\left|\Theta_{\mathcal{M}}^{\lambda,\pi} - (t_1 - t_0)\right| \geq \delta\right] = 0
$$

when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(0)$.

The process $(\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t\geq 0, i\in\mathbb{Z}}$ defined in Lemma II.9.2 is closely related to the process defined in Lemma II.8.2. If $t_0 = 0$, then the process starts from a vacant initial situation and a match falls on $\lfloor \mathbf{n}_\lambda x_1 \rfloor$ at time $\mathbf{a}_\lambda t_1$. It does not depend on $x_0 \in \mathbb{R}$. Since

$0 < t_1 < 1$, the zone $(x_1)_\lambda$ is not completely filled at time $\mathbf{a}_\lambda(t_1 + \varkappa_{\lambda,\pi})$, see Lemma II.9.1-1 (using space stationarity). The process is then governed by the propagation processes $(N_t^P(i))_{t\geq 0, i\in\mathbb{Z}}$ and the seed processes $(N_t^S(i))_{t\geq 0, i\in\mathbb{Z}}$ with the same rules as the $(\lambda,\pi)-$FFP. As seen in **Micro**(0) in Subsection II.4.4, the fire is extinguish at time $\mathbf{a}_\lambda(t_1 + \varkappa_{\lambda,\pi})$.

If $t_0 > 1$, then the process starts at time 0 from an occupied initial situation, nothing happens until a match falls in $\lfloor \mathbf{n}_\lambda x_0 \rfloor \in I_A^\lambda$ at time $\mathbf{a}_\lambda t_0$. Two fires start: one goes to the left and one goes to the right. Thus, on $\Omega_{\lambda,\pi}^{P,2A,2A}(x_0, t_0)$, recall Definition II.4.7, each site of I_A^λ burns and extinguishes before $\mathbf{a}_\lambda(t_0 + \varkappa_{\lambda,\pi})$, recall **Macro**(0) in Subsection II.4.4. Hence, the zone $(x_1)_\lambda$ is not completely filled when the match falls on $\lfloor \mathbf{n}_\lambda x_1 \rfloor$ at time $\mathbf{a}_\lambda t_1$, see Lemma II.9.1-1, because $\mathbf{a}_\lambda(t_0 + \varkappa_{\lambda,\pi}) < \mathbf{a}_\lambda t_1 < \mathbf{a}_\lambda(t_0 + 1)$ for all (λ,π) sufficiently close to the regime $\mathcal{R}(0)$.

Proof. The proof is very similar to the proof of Lemma II.8.2. We first define the simplest process with an instantaneous propagation: if a match falls in a cluster, it destroys instantaneously the entire connected component. Secondly, we flank the killed cluster $C^P((\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t\geq 0, i\in\mathbb{Z}}, (x_1, t_1))$ to estimate the time needed to become again occupied.

Without loss of generality, we assume that $x_1 = 0$ and $x_0 \in [-A\,, A]$ (using space stationarity).

Step 1. Let $\tau_0 < \tau_1 < \tau_0 + 1$ be fixed. Put $\vartheta_{\tau_0,t}^\lambda(i) = \min(N_{\mathbf{a}_\lambda(\tau_0+t)}^S(i) - N_{\mathbf{a}_\lambda\tau_0}^S(i), 1)$ and $\vartheta_{\tau_1,t}^\lambda(i) = \min(N_{\mathbf{a}_\lambda(\tau_1+t)}^S(i) - N_{\mathbf{a}_\lambda\tau_1}^S(i), 1)$ for all $t > 0$ and all $i \in \mathbb{Z}$. We define the time needed for the destroyed cluster to be fully regenerated

$$\Xi_{\tau_0,\tau_1}^\lambda = \inf\left\{t > 0 : \forall i \in C(\vartheta_{\tau_0,\tau_1-\tau_0}^\lambda, 0),\ \vartheta_{\tau_1,t}^\lambda(i) = 1\right\}.$$

Then for all $\delta > 0$,

$$\lim_{\lambda\to 0} \mathbb{P}\left[|\Xi_{\tau_0,\tau_1}^\lambda - (\tau_1 - \tau_0)| \geq \delta\right] = 0.$$

This has been checked in Step 1 in the proof of Lemma II.8.2.

Step 2. Assume $t_0 = 0$. In that case, the process does not depend on x_0. Consider the event $\Omega_{\lambda,\pi}^{P,2A,2A}(0, t_1)$, recall Definition II.4.7. We define

$$\begin{aligned}\tilde\Omega_{\lambda,\pi}^{P,A,\mathcal{M}} = \Omega_{\lambda,\pi}^{P,2A,2A}(0, t_1) \cap \{\exists i_1 \in [\![-\mathbf{m}_\lambda\,, 0]\!], N_{\mathbf{a}_\lambda(t_1+\varkappa_{\lambda,\pi})}^S(i_1) = 0\}\\ \cap \{\exists i_2 \in [\![0\,, \mathbf{m}_\lambda]\!], N_{\mathbf{a}_\lambda(t_1+\varkappa_{\lambda,\pi})}^S(i_2) = 0\}.\end{aligned}$$

Lemma II.4.3 together with Lemma II.9.1-1 show that $\mathbb{P}\left[\tilde\Omega_{\lambda,\pi}^{P,A,\mathcal{M}}\right]$ tends to 1 when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(0)$ (because $t_1 + \varkappa_{\lambda,\pi} < (t_1+1)/2 < 1$ for all (λ,π) sufficiently close to the regime $\mathcal{R}(0)$).

Next, on $\tilde\Omega_{\lambda,\pi}^{P,A,\mathcal{M}}(0, t_1)$, there holds that

$$C(\vartheta_{0,t_1+\kappa_{\lambda,\pi}}^\lambda, 0) := [\![C^-\,, C^+]\!] \subset [\![i_1\,, i_2]\!] \subset [\![-\mathbf{m}_\lambda\,, \mathbf{m}_\lambda]\!].$$

Since, by definition, no seed falls on C^+ and on C^- until $\mathbf{a}_\lambda(t_1+\varkappa_{\lambda,\pi})$ and since we start from a vacant initial situation, we also deduce that

$$\zeta_t^{\lambda,\pi,\mathcal{M}}(C^-)=\zeta_t^{\lambda,\pi,\mathcal{M}}(C^+)=0$$

for all $t\in[0\,,\mathbf{a}_\lambda(t_1+\varkappa_{\lambda,\pi})]\supset[\mathbf{a}_\lambda t_1\,,\mathbf{a}_\lambda(t_1+\varkappa_{\lambda,\pi})]$. As seen in **Micro**(0) in Subsection II.4.4, the match falling on 0 at time $\mathbf{a}_\lambda t_1$ destroys exactly $C^P((\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t\geq 0,i\in\mathbb{Z}},(0,t_1))$ and

$$C^P((\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t\geq 0,i\in\mathbb{Z}},(0,t_1))\subset[\![C^-\,,C^+]\!]\subset[\![-\mathbf{m}_\lambda\,,\mathbf{m}_\lambda]\!]$$

with $\zeta^{\lambda,\pi,\mathcal{M}}_{\mathbf{a}_\lambda(t_1+\varkappa_{\lambda,\pi})}(i)\leq 1$ for all $i\in\mathbb{Z}$ (the fire is extinguished at time $\mathbf{a}_\lambda(t_1+\varkappa_{\lambda,\pi})$).

Since $C^P((\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t\geq 0,i\in\mathbb{Z}},(0,t_1))$ clearly contains $C(\vartheta^\lambda_{0,t_1},0)$, we deduce that, on $\tilde\Omega^{P,A,\mathcal{M}}_{\lambda,\pi}$,

$$t_1+\Xi^\lambda_{0,t_1}\leq t_1+\Theta^{\lambda,\pi}_{\mathcal{M}}\leq t_1+\varkappa_{\lambda,\pi}+\Xi^\lambda_{0,t_1+\varkappa_{\lambda,\pi}}.$$

Remark now that the function $: t\mapsto t+\Xi^\lambda_{0,t}$ is a.s. non decreasing and right-continuous. We thus deduce from Step 1 that

$$t_1+\Theta^{\lambda,\pi}_{\mathcal{M}}\xrightarrow[\lambda,\pi]{\mathbb{P}}2t_1$$

in probability, whence for all $\delta>0$ and all $\varepsilon>0$, there holds that $\mathbb{P}\left[\left|\Theta^{\lambda,\pi}_{\mathcal{M}}-t_1\right|\geq\delta\right]<\varepsilon$ for all (λ,π) sufficiently close to the regime $\mathcal{R}(0)$.

Step 3. Assume now $t_0>1$. We may and will assume $x_0\in(-A\,,0)$, by symmetry.

Consider the events $\Omega^{P,2A,2A}_{\lambda,\pi}(x_0,t_0)$ and $\Omega^{P,2A,2A}_{\lambda,\pi}(0,t_1)$, recall Definition II.4.7. We define

$$\begin{aligned}\tilde\Omega^{P,A,\mathcal{M}}_{\lambda,\pi}:=\Omega^{P,2A,2A}_{\lambda,\pi}(0,t_1)\cap\Omega^{P,2A,2A}_{\lambda,\pi}(x_0,t_0)&\\ \cap\{\exists i_1\in[\![-\mathbf{m}_\lambda\,,0]\!],N^S_{\mathbf{a}_\lambda(t_1+\varkappa_{\lambda,\pi})}(i_1)-N^S_{\mathbf{a}_\lambda t_0}(i_1)=0\}&\\ \cap\{\exists i_2\in[\![0\,,\mathbf{m}_\lambda]\!],N^S_{\mathbf{a}_\lambda(t_1+\varkappa_{\lambda,\pi})}(i_2)-N^S_{\mathbf{a}_\lambda t_0}(i_2)=0\}&.\end{aligned}$$

Lemma II.4.3 together with Lemma II.9.1-1 directly imply that $\mathbb{P}\left[\tilde\Omega^{P,A,\mathcal{M}}_{\lambda,\pi}\right]$ tends to 1 when $\lambda\to 0$ and $\pi\to\infty$ in the regime $\mathcal{R}(0)$ (because $t_1+\varkappa_{\lambda,\pi}-t_0<(t_1-t_0+1)/2<1$ for all (λ,π) sufficiently close to the regime $\mathcal{R}(0)$).

First, since the sites $\lfloor\mathbf{n}_\lambda A\rfloor+1$ and $-\lfloor\mathbf{n}_\lambda A\rfloor-1$ remain vacant all the time and since I^λ_A is completely occupied at time $\mathbf{a}_\lambda t_0$, on $\Omega^{P,2A,2A}_{\lambda,\pi}(x_0,t_0)$, as seen in **Macro**(0) in Subsection II.4.4, the match falling on $\lfloor\mathbf{n}_\lambda x_0\rfloor$ at time $\mathbf{a}_\lambda t_0$ destroys each site of I^λ_A during the time interval $[\mathbf{a}_\lambda t_0\,,\mathbf{a}_\lambda(t_0+\varkappa_{\lambda,\pi})]$. Furthermore, there is no more burning tree in I^λ_A at time $\mathbf{a}_\lambda(t_0+\varkappa_{\lambda,\pi})$.

Next, on $\tilde\Omega^{P,A,\mathcal{M}}_{\lambda,\pi}$, since no seed falls on i_1 and i_2 during the time interval $[\mathbf{a}_\lambda t_0\,,\mathbf{a}_\lambda(t_1+\varkappa_{\lambda,\pi})]$, we clearly have

$$C(\vartheta^\lambda_{t_0,t_1+\varkappa_{\lambda,\pi}},0):=[\![C^-\,,C^+]\!]\subset[\![i_1\,,i_2]\!]\subset[\![-\mathbf{m}_\lambda\,,\mathbf{m}_\lambda]\!].$$

Since, by definition, no seed falls on C^- and on C^+ during $[\mathbf{a}_\lambda t_0\,,\mathbf{a}_\lambda(t_1+\varkappa_{\lambda,\pi})]$ and since C^- and C^+ are made vacant during the time interval $[\mathbf{a}_\lambda t_0\,,\mathbf{a}_\lambda(t_0+\varkappa_{\lambda,\pi})]$, we deduce that

$$\zeta^{\lambda,\pi,\mathcal{M}}_{\mathbf{a}_\lambda t}(C^-)=\zeta^{\lambda,\pi,\mathcal{M}}_{\mathbf{a}_\lambda t}(C^+)=0 \text{ for all } t\in[t_1\,,t_1+\varkappa_{\lambda,\pi}].$$

Hence, as seen in **Micro**(0) in Subsection II.4.4, the match falling on 0 at time $\mathbf{a}_\lambda t_1$ destroys exactly the zone $C^P((\zeta^{\lambda,\pi,\mathcal{M}}_t(i))_{t\geq 0,i\in\mathbb{Z}},(0,t_1))\subset[\![C^-\,,C^+]\!]\subset[\![i_1\,,i_2]\!]$.

To summarize, since $C^P((\zeta^{\lambda,\pi,\mathcal{M}}_t(i))_{t\geq 0,i\in\mathbb{Z}},(0,t_1))$ clearly contains $C(\vartheta^\lambda_{t_0+\varkappa_{\lambda,\pi},t_1},0)$, on $\tilde\Omega^{P,A,\mathcal{M}}_{\lambda,\pi}$, we have

$$C(\vartheta^\lambda_{t_0+\varkappa_{\lambda,\pi},t_1},0)\subset C^P((\zeta^{\lambda,\pi,\mathcal{M}}_t(i))_{t\geq 0,i\in\mathbb{Z}},(0,t_1))\subset C(\vartheta^\lambda_{t_0,t_1+\varkappa_{\lambda,\pi}},0)\subset[\![i_1\,,i_2]\!]$$

with additionally $\zeta^{\lambda,\pi,\mathcal{M}}_{\mathbf{a}_\lambda(t_1+\varkappa_{\lambda,\pi})}(i)\leq 1$ for all $i\in I^\lambda_A$.

We deduce that, on $\tilde\Omega^{P,A,\mathcal{M}}_{\lambda,\pi}$ and for all (λ,π) sufficiently close to the regime $\mathcal{R}(0)$,

$$t_1+\Xi^\lambda_{t_0+\varkappa_{\lambda,\pi},t_1}\leq t_1+\Theta^{\lambda,\pi}_{\mathcal{M}}\leq t_1+\varkappa_{\lambda,\pi}+\Xi^\lambda_{t_0,t_1+\varkappa_{\lambda,\pi}}.$$

Then, one easily concludes. The function $s\mapsto t_1+\Xi^\lambda_{t_0+s,t_1}$ is a.s. non increasing and right-continuous, while the function $s\mapsto t_1+s+\Xi^\lambda_{t_0,t_1+s}$ is a.s. non decreasing and right-continuous. We thus deduce from Step 1 that

$$t_1+\Theta^{\lambda,\pi}_{\mathcal{M}}\xrightarrow[\lambda,\pi]{\mathbb{P}}2t_1-t_0,$$

as desired. □

II.9.3. Persistent effect of microscopic fires

Here we study the effect of microscopic fires. First, they produce a barrier, and then, if there are alternatively macroscopic fires on the left and right, they still have an effect. This phenomenon is illustrated on Figure II.10 in the case of the limit process.

We say that $\mathcal{P}=(\varepsilon;(x_0,t_0),(x_1,t_1),\dots,(x_K,t_K))$ satisfies (PP) if

1. $K\geq 2$ and $\varepsilon\in\{-1,1\}$;
2. $t_0\in\{0\}\cup(1\,,\infty)$ and $t_0<t_1<t_2<\cdots<t_K$;
3. for all $k=0,\dots,K-1$, $t_{k+1}-t_k<1$;
4. $t_2-t_0>1$ and for all $k=2,\dots,K-2$, $t_{k+2}-t_k>1$;
5. for all $k=0,\dots,K$, $x_k\in(-A\,,A)$ and for all $k=2,\dots,K$, $\varepsilon_k(x_k-x_1)>0$, where we set $\varepsilon_k=(-1)^k\varepsilon$.

Let $\mathcal{P}$ satisfy (PP). Consider two Poisson processes $(N_t^S(i))_{t\geq 0, i\in\mathbb{Z}}$ and $(N_t^P(i))_{t\geq 0, i\in\mathbb{Z}}$ with respective rates 1 and π, all these processes being independent. We define the process $(\zeta_t^{\lambda,\pi,\mathcal{P}}(i))_{t\geq 0, i\in I_A^\lambda}$ as follows

$$
\begin{aligned}
\zeta_t^{\lambda,\pi,\mathcal{P}}(i) =& (1+\mathbf{1}_{\{i=\lfloor \mathbf{n}_\lambda x_0\rfloor, t\geq \mathbf{a}_\lambda t_0\}})\mathbf{1}_{\{t_0\geq 1\}} + \mathbf{1}_{\{i=\lfloor \mathbf{n}_\lambda x_1\rfloor, t\geq \mathbf{a}_\lambda t_1, \zeta^{\lambda,\pi,\mathcal{P}}_{\mathbf{a}_\lambda t_1 -}(\lfloor \mathbf{n}_\lambda x_1\rfloor)=1\}} \\
&+ \sum_{k=2}^{K} \mathbf{1}_{\{i=\lfloor \mathbf{n}_\lambda x_k\rfloor, t\geq \mathbf{a}_\lambda t_k, \zeta^{\lambda,\pi,\mathcal{P}}_{\mathbf{a}_\lambda t_k -}(\lfloor \mathbf{n}_\lambda x_k\rfloor)=1\}} \\
&+ \int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda,\pi,\mathcal{P}}(i)=0\}}\,\mathrm{d}N_s^S(i) \\
&+ \int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda,\pi,\mathcal{P}}(i-1)=2, \zeta_{s-}^{\lambda,\pi,\mathcal{P}}(i)=1\}}\,\mathrm{d}N_s^P(i-1) \\
&+ \int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda,\pi,\mathcal{P}}(i+1)=2, \zeta_{s-}^{\lambda,\pi,\mathcal{P}}(i)=1\}}\,\mathrm{d}N_s^P(i+1) \\
&- 2\int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda,\pi,\mathcal{P}}(i)=2\}}\,\mathrm{d}N_s^P(i)
\end{aligned}
$$

with the convention $\zeta_t^{\lambda,\pi,\mathcal{P}}(\lfloor \mathbf{n}_\lambda A\rfloor+1) = \zeta_t^{\lambda,\pi,\mathcal{P}}(-\lfloor \mathbf{n}_\lambda A\rfloor-1) = 0$ for all $t\in[0,\infty)$.
We now explain the behaviour of the process $(\zeta_t^{\lambda,\pi,\mathcal{P}}(i))_{t\geq 0, i\in I_A^\lambda}$.

- If $t_0 = 0$, then the process starts from a vacant initial configuration. The match falling on $\lfloor \mathbf{n}_\lambda x_1\rfloor$ at time $\mathbf{a}_\lambda t_1 \in (0, \mathbf{a}_\lambda)$ creates a barrier, see Lemma II.9.2, because $t_1\in(0,1)$. Then, fires start in $\lfloor \mathbf{n}_\lambda x_k\rfloor$ alternately on the right and on the left of $\lfloor \mathbf{n}_\lambda x_1\rfloor$ at times $\mathbf{a}_\lambda t_k$ for all $k=2,\dots,K$ and fires spread accross $\mathbb{Z}$ according to the same rules as the $(\lambda,\pi,A)-$FFP.
- If $t_0 > 1$, the process starts from an occupied initial situation. Nothing happens until a match falls in $\lfloor \mathbf{n}_\lambda x_0\rfloor$ at time $\mathbf{a}_\lambda t_0$ and spreads across I_A^λ (because all the sites are occupied at time $\mathbf{a}_\lambda t_0-$ and $\lfloor \mathbf{n}_\lambda A\rfloor+1$ and $-\lfloor \mathbf{n}_\lambda A\rfloor-1$ are vacants). Next, a match falls on $\lfloor \mathbf{n}_\lambda x_1\rfloor$ at time $\mathbf{a}_\lambda t_1 \in (\mathbf{a}_\lambda t_0, \mathbf{a}_\lambda(t_0+1))$. It then creates a barrier, see Lemma II.9.2. Afterwards, matches fall successively in $\lfloor \mathbf{n}_\lambda x_k\rfloor$ at times $\mathbf{a}_\lambda t_k$ for each $k=2,\dots,K$ and fires spread accross I_A^λ according to the same rules as the $(\lambda,\pi,A)-$FFP.

Consider the event

$$
\Omega_{\mathcal{P}}^{S,P}(\lambda,\pi) = \{\forall k\in\{2,\dots,K\}, \exists j\in (x_1)_\lambda, \forall t\in[t_k+\varkappa_{\lambda,\pi}, t_k+1], \zeta_{\mathbf{a}_\lambda t}^{\lambda,\pi,\mathcal{P}}(j)=0\}.
$$

Lemma II.9.3. *Let $\mathcal{P} = (\varepsilon; (x_0,t_0), (x_1,t_1), \dots, (x_K,t_K))$ satisfy (PP). For each $\lambda\in(0,1)$ and each $\pi\geq 1$, consider the process $(\zeta_t^{\lambda,\pi,\mathcal{P}}(i))_{t\geq 0, i\in\mathbb{Z}}$ defined above.*
If $t_2 - t_1 < t_1 - t_0$, when $\lambda\to 0$ and $\pi\to\infty$ in the regime $\mathcal{R}(0)$, there holds

$$
\lim_{\lambda,\pi} \mathbb{P}\left[\Omega_{\mathcal{P}}^{S,P}(\lambda,\pi)\right] = 1.
$$

Proof. Without loss of generality, we assume $x_1 = 0$ and $(x_k)_{k=0,2,\dots,K} \subset [-A\,,A]$.
We define, recall Definition II.4.7,

$$\Omega^{P,A,\mathcal{P}}_{\lambda,\pi} = \Omega^{P,2A,2A}_{\lambda,\pi}(0,t_1) \cap \bigcap_{k=0,2,\dots,K} \Omega^{P,2A,2A}_{\lambda,\pi}(x_k,t_k).$$

There holds that $\mathbb{P}\left[\Omega^{P,A,\mathcal{P}}_{\lambda,\pi}\right]$ tends to 1 as $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(0)$ by Lemma II.4.3. In the whole proof, we work on $\Omega^{P,A,\mathcal{P}}_{\lambda,\pi}$ and assume that (λ,π) is sufficiently close to the regime $\mathcal{R}(0)$ in such a way that $\varkappa_{\lambda,\pi} < \min_{i\neq j}|t_i - t_j|$ and $\min_{k=0,2,\dots,K}|\lfloor \mathbf{n}_\lambda x_k\rfloor| \geq \mathbf{m}_\lambda$.

For simplicity, we assume that $\varepsilon = -1$, $t_0 = 0$ and that K is even. The other cases are treated similarly (see for example Lemma II.9.2). Fix $\alpha = 1/K$. We define $\mathcal{M} := ((0,0),(0,t_1))$, recall Lemma II.9.2.

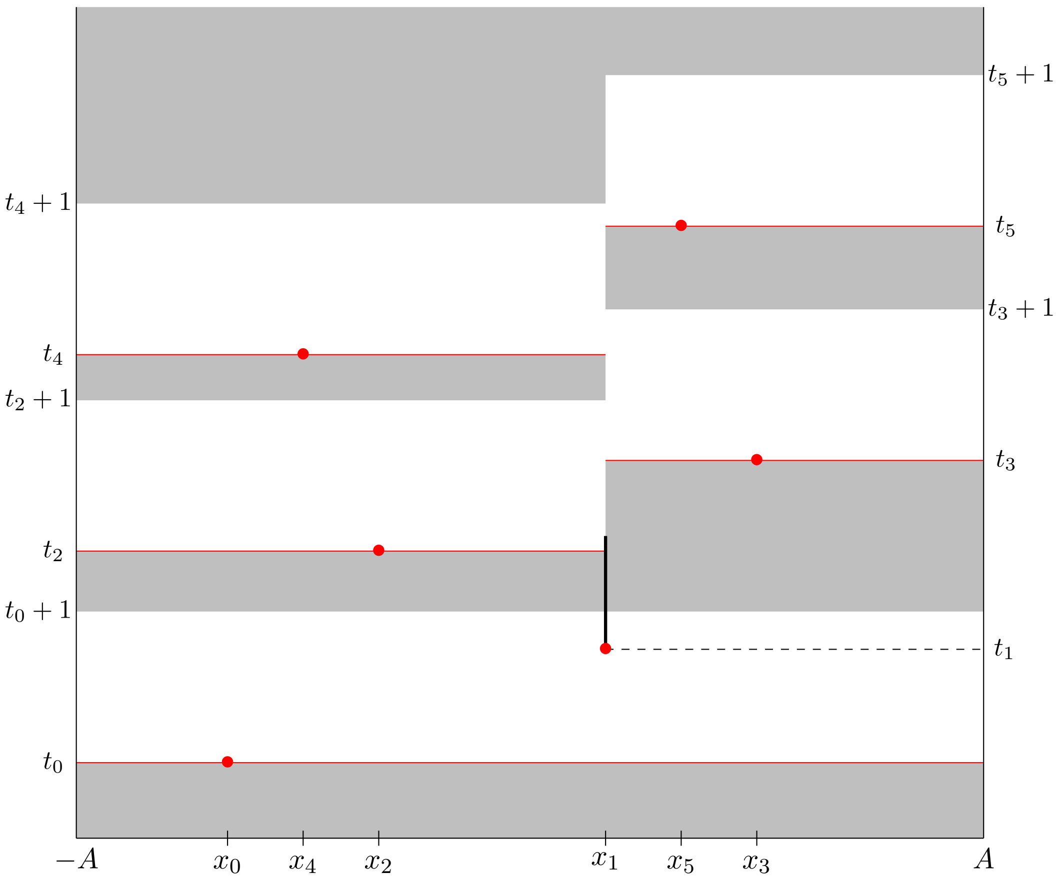

Figure II.10.: Persistent effect of microscopic fires.

Here $\mathcal{P} = (-1;(x_0,t_0),(x_1,t_1),(x_2,t_2),(x_3,t_3),(x_4,t_4),(x_5,t_5))$.

Since $\lfloor \mathbf{n}_\lambda A\rfloor + 1$ and $-\lfloor \mathbf{n}_\lambda A\rfloor - 1$ remain vacant all the time, on $\Omega^{P,A,\mathcal{P}}_{\lambda,\pi}$, a burning tree at time $\mathbf{a}_\lambda t$ is either a front of a fire or has vacant neighbors. Thus, there is no burning tree outside $\cup_{k=1,\dots,K}[\mathbf{a}_\lambda t_k \,, \mathbf{a}_\lambda(t_k + \varkappa_{\lambda,\pi})]$.

First fire. We put $C^P := C^P((\zeta_t^{\lambda,\pi,\mathcal{P}}(i))_{t\geq 0, i\in\mathbb{Z}}, (0, t_1))$, the destroyed cluster, recall (II.4.14). Since $t_1 + \varkappa_{\lambda,\pi} < 1$, $C^P \subset [\![-\lfloor \alpha \mathbf{m}_\lambda \rfloor \,, \lfloor \alpha \mathbf{m}_\lambda \rfloor]\!]$ with probability tending to 1 (use Lemma II.9.1-1, space/time stationarity and **Micro**(0) in Subsection II.4.4). Thus the match falling at time $\mathbf{a}_\lambda t_1$ destroys nothing outside $[\![-\lfloor \alpha \mathbf{m}_\lambda \rfloor \,, \lfloor \alpha \mathbf{m}_\lambda \rfloor]\!]$ and there is no more burning tree in I_A^λ at time $\mathbf{a}_\lambda(t_1 + \varkappa_{\lambda,\pi})$.

Second fire. Since $t_2 > 1$, at least one seed has fallen, during $[0 \,, \mathbf{a}_\lambda t_2)$, on each site of $[\![-\lfloor \mathbf{n}_\lambda A \rfloor \,, -\lfloor \alpha \mathbf{m}_\lambda \rfloor - 1]\!]$ with probability tending to 1 (use Lemma II.9.1-4 and space/time stationarity). Since this zone has not been affected by a fire during the time interval $[0 \,, \mathbf{a}_\lambda t_2)$, this zone is completely occupied at time $\mathbf{a}_\lambda t_2-$.

Besides, with probability tending to 1, there is (at least) an empty site in $C^P \subset [\![-\lfloor \alpha \mathbf{m}_\lambda \rfloor \,, \lfloor \alpha \mathbf{m}_\lambda \rfloor]\!]$ during the time interval $(\mathbf{a}_\lambda(t_1 + \varkappa_{\lambda,\pi}) \,, \mathbf{a}_\lambda(t_2 + \varkappa_{\lambda,\pi}))$ because $t_1 + \varkappa_{\lambda,\pi} < t_2 < t_2 + \varkappa_{\lambda,\pi} < t_1 + \Theta_{\mathcal{M}}^{\lambda,\pi}$ with probability tending to 1 (by Lemma II.9.2, $\Theta_{\mathcal{M}}^{\lambda,\pi} \simeq t_1 - t_0 = t_1$ and $t_2 - t_1 < t_1 - t_0 = t_1$ by assumption) and because by definition of $\Theta_{\mathcal{M}}^{\lambda,\pi}$, there is an empty site in $C^P \subset [\![-\lfloor \alpha \mathbf{m}_\lambda \rfloor \,, \lfloor \alpha \mathbf{m}_\lambda \rfloor]\!]$ during $[\mathbf{a}_\lambda(t_1 + \varkappa_{\lambda,\pi}) \,, \mathbf{a}_\lambda(t_1 + \Theta_{\mathcal{M}}^{\lambda,\pi})]$.

Thus, the fire ignited on $\lfloor \mathbf{n}_\lambda x_2 \rfloor \in [\![-\lfloor \mathbf{n}_\lambda A \rfloor \,, -\mathbf{m}_\lambda]\!]$ at time $\mathbf{a}_\lambda t_2$ burns each site of the zone $[\![-\lfloor \mathbf{n}_\lambda A \rfloor \,, -\lfloor \alpha \mathbf{m}_\lambda \rfloor - 1]\!]$ before $\mathbf{a}_\lambda(t_2 + \varkappa_{\lambda,\pi})$ and does not affect the zone $[\![\lfloor \alpha \mathbf{m}_\lambda \rfloor + 1 \,, \lfloor \mathbf{n}_\lambda A \rfloor]\!]$, thanks to $\Omega_{\lambda,\pi}^{P,2A,2A}(x_2, t_2)$, as seen in **Macro**(0) in Subsection II.4.4.

Third fire. All the sites of $[\![\lfloor \alpha \mathbf{m}_\lambda \rfloor + 1 \,, \lfloor \mathbf{n}_\lambda A \rfloor]\!]$ are occupied at time $\mathbf{a}_\lambda t_3-$ with probability tending to 1 (because on $\Omega_{\lambda,\pi}^{P,2A,2A}(0, t_1) \cap \Omega_{\lambda,\pi}^{P,2A,2A}(x_2, t_2)$, they have not been affected by a fire during $[0 \,, \mathbf{a}_\lambda t_3)$ and because $t_3 > t_2 > 1$, see Lemma II.9.1-4).

Next, since $t_3 - t_2 < 1$, the probability that there is a site in $[\![-\lfloor 2\alpha \mathbf{m}_\lambda \rfloor \,, -\lfloor \alpha \mathbf{m}_\lambda \rfloor - 1]\!]$ where no seed falls during $[\mathbf{a}_\lambda t_2 \,, \mathbf{a}_\lambda(t_2 + 1))$ tends to 1 as $\lambda \to 0$ (use Lemma II.9.1-1 and space/time stationarity). Thus, with probability tending to 1, there exists a vacant site in $[\![-\lfloor 2\alpha \mathbf{m}_\lambda \rfloor \,, -\lfloor \alpha \mathbf{m}_\lambda \rfloor]\!]$ during $[\mathbf{a}_\lambda(t_2 + \varkappa_{\lambda,\pi}) \,, \mathbf{a}_\lambda(t_2 + 1)) \supset [\mathbf{a}_\lambda t_3 \,, \mathbf{a}_\lambda(t_3 + \varkappa_{\lambda,\pi})]$ (because all the sites of $[\![-\lfloor \mathbf{n}_\lambda A \rfloor \,, -\lfloor \alpha \mathbf{m}_\lambda \rfloor - 1]\!]$ have been made vacant by the fire 2).

Thus, the fire ignited on $\lfloor \mathbf{n}_\lambda x_3 \rfloor \in [\![\mathbf{m}_\lambda \,, \lfloor \mathbf{n}_\lambda A \rfloor]\!]$ at time $\mathbf{a}_\lambda t_3$ burns each site of $[\![\lfloor \alpha \mathbf{m}_\lambda \rfloor + 1 \,, \lfloor \mathbf{n}_\lambda A \rfloor]\!]$ before $\mathbf{a}_\lambda(t_3 + \varkappa_{\lambda,\pi})$ and does not affect the zone $[\![-\lfloor \mathbf{n}_\lambda A \rfloor \,, -\lfloor 2\alpha \mathbf{m}_\lambda \rfloor]\!]$ with probability tending to 1, thanks to $\Omega_{\lambda,\pi}^{P,2A,2A}(x_3, t_3)$, as seen in **Macro**(0) in Subsection II.4.4.

Fourth fire. All the sites of $[\![-\lfloor \mathbf{n}_\lambda A \rfloor \,, -\lfloor 2\alpha \mathbf{m}_\lambda \rfloor - 1]\!]$ are occupied at time $\mathbf{a}_\lambda t_4-$ with probability tending to 1 (because on $\Omega_{\lambda,\pi}^{P,2A,2A}(0, t_1) \cap \Omega_{\lambda,\pi}^{P,2A,2A}(x_2, t_2) \cap \Omega_{\lambda,\pi}^{P,2A,2A}(x_3, t_3)$, they have not been affected by a fire during $(\mathbf{a}_\lambda(t_2 + \varkappa_{\lambda,\pi}) \,, \mathbf{a}_\lambda t_4)$ and because $t_4 - t_2 - \varkappa_{\lambda,\pi} > 1$, see Lemma II.9.1-4 and space/time stationarity).

Since $t_4 - t_3 < 1$, the probability that there is a site in $[\![\lfloor \alpha \mathbf{m}_\lambda \rfloor + 1 \,, \lfloor 2\alpha \mathbf{m}_\lambda \rfloor]\!]$ where no seed falls during $[\mathbf{a}_\lambda t_3 \,, \mathbf{a}_\lambda(t_3 + 1))$ tends to 1 as $\lambda \to 0$ (use Lemma II.9.1-1 and space/time stationarity). Hence there is at least one vacant site in $[\![\lfloor \alpha \mathbf{m}_\lambda \rfloor + 1 \,, \lfloor 2\alpha \mathbf{m}_\lambda \rfloor]\!]$

during $[\mathbf{a}_\lambda(t_3+\varkappa_{\lambda,\pi})\,,\mathbf{a}_\lambda(t_3+1))\supset[\mathbf{a}_\lambda t_4\,,\mathbf{a}_\lambda(t_4+\varkappa_{\lambda,\pi})]$, with probability tending to 1 (because all the sites of $[\![\lfloor\alpha\mathbf{m}_\lambda\rfloor+1\,,\lfloor\mathbf{n}_\lambda A\rfloor]\!]$ have been made vacant by the fire 3).

Thus, the fire ignited on $\lfloor\mathbf{n}_\lambda x_4\rfloor\in[\![-\lfloor\mathbf{n}_\lambda A\rfloor\,,-\mathbf{m}_\lambda]\!]$ at time $\mathbf{a}_\lambda t_4$ burns each site of the zone $[\![-\lfloor\mathbf{n}_\lambda A\rfloor\,,-\lfloor 2\alpha\mathbf{m}_\lambda\rfloor-1]\!]$ before $\mathbf{a}_\lambda(t_4+\varkappa_{\lambda,\pi})$ and does not affect the zone $[\![\lfloor 2\alpha\mathbf{m}_\lambda\rfloor+1\,,\lfloor\mathbf{n}_\lambda A\rfloor]\!]$ with probability tending to 1, thanks to $\Omega^{P,2A,2A}_{\lambda,\pi}(x_4,t_4)$, as seen in $\mathbf{Macro}(0)$ in Subsection II.4.4.

Last fire and conclusion. Iterating the procedure, we see that with probability tending to 1 as $\lambda\to 0$ and $\pi\to\infty$ in the regime $\mathcal{R}(0)$, the zone

$$[\![-\lfloor\mathbf{n}_\lambda A\rfloor\,,-\lfloor(K\alpha/2)\mathbf{m}_\lambda\rfloor-1]\!]=[\![-\lfloor\mathbf{n}_\lambda A\rfloor\,,-\lfloor\mathbf{m}_\lambda/2\rfloor-1]\!]$$

is completely occupied at time $\mathbf{a}_\lambda t_K-$ and there is at least one vacant site in $[\![\lfloor(K-1)\alpha/2\mathbf{m}_\lambda\rfloor+1\,,\lfloor\mathbf{m}_\lambda/2\rfloor]\!]$ during the time interval $[\mathbf{a}_\lambda(t_{K-1}+\varkappa_{\lambda,\pi})\,,\mathbf{a}_\lambda(t_{K-1}+1))\supset[\mathbf{a}_\lambda t_K\,,\mathbf{a}_\lambda(t_K+\varkappa_{\lambda,\pi})]$. Thus, the fire ignited on $\lfloor\mathbf{n}_\lambda x_K\rfloor\in[\![-\lfloor\mathbf{n}_\lambda A\rfloor\,,-\mathbf{m}_\lambda]\!]$ at time $\mathbf{a}_\lambda t_K$ destroys each site of the zone $[\![-\lfloor\mathbf{n}_\lambda A\rfloor\,,-\lfloor\mathbf{m}_\lambda/2\rfloor-1]\!]$ before $\mathbf{a}_\lambda(t_K+\varkappa_{\lambda,\pi})$ and does not affect the zone $[\![\lfloor\mathbf{m}_\lambda/2\rfloor\,,\lfloor\mathbf{n}_\lambda A\rfloor]\!]$.

Finally, the probability that there is at least one site in $[\![-\mathbf{m}_\lambda\,,-\mathbf{m}_\lambda/2]\!]$ with no seed falling during $[\mathbf{a}_\lambda t_K\,,\mathbf{a}_\lambda(t_K+1)]$ tends to 1 (by Lemma II.9.1-1). Consequently, the probability that there is a vacant site in $[\![-\mathbf{m}_\lambda\,,-\lfloor\mathbf{m}_\lambda/2\rfloor]\!]$ during $[\mathbf{a}_\lambda(t_K+\varkappa_{\lambda,\pi})\,,\mathbf{a}_\lambda(t_K+1)]$ tends to 1. All this implies the claim. $\square$

II.9.4. Heart of the proof

II.9.4.1. The coupling

We are going to construct a coupling between the $(\lambda,\pi,A)-$FFP (on the time interval $[0\,,\mathbf{a}_\lambda T]$) and the $A-$LFFP(0) (on $[0\,,T]$). Let π_M be a Poisson measure on $\mathbb{R}\times[0\,,\infty)$ with intensity measure $\mathrm{d}x\,\mathrm{d}t$.

First, we take for the matches of the discrete process the Poisson processes

$$N^M_t(i)=\pi_M([i/\mathbf{n}_\lambda\,,(i+1)/\mathbf{n}_\lambda)\times[0\,,t/\mathbf{a}_\lambda])$$

for all $i\in\mathbb{Z}$ and $t\in[0\,,T]$.

We call $n:=\pi_M([0,T]\times[-A\,,A])$ and we consider the marks $(T_q,X_q)_{q=1,\dots,n}$ of π_M ordered in such a way that $0<T_1<\cdots<T_n<T$.

Next, we introduce two families of i.i.d. Poisson processes $(N^S_t(i))_{t\geq 0,i\in\mathbb{Z}}$ and $(N^P_t(i))_{t\geq 0,i\in\mathbb{Z}}$ with respective parameters 1 and π, independent of π_M.

The $(\lambda,\pi,A)-$FFP $(\eta^{\lambda,\pi}_t(i))_{t\geq 0,i\in I^\lambda_A}$ is built from the seed processes $(N^S_t(i))_{t\geq 0,i\in\mathbb{Z}}$, the match processes $(N^M_t(i))_{t\geq 0,i\in\mathbb{Z}}$ and the propagation processes $(N^P_t(i))_{t\geq 0,i\in\mathbb{Z}}$.

Finally, we build the $A-$LFFP(0) $(Z_t(x),H_t(x),F_t(x))_{t\in[0,T],x\in[-A,A]}$ from π_M and observe that it is independent of $(N^S_t(i))_{t\in[0,\mathbf{a}_\lambda T],i\in\mathbb{Z}}$ and $(N^P_t(i))_{t\in[0,\mathbf{a}_\lambda T],i\in\mathbb{Z}}$.

Observe that if a match falls on some X_q at time T_q for the $A-$LFFP(0), it also falls on $\lfloor\mathbf{n}_\lambda X_q\rfloor$ at time $\mathbf{a}_\lambda T_q$ in the discrete process.

II.9.4.2. A favorable event

We set $T_0 = 0$ and introduce

$$\mathcal{T}_M = \{T_0, T_1, \dots, T_n\} \text{ and } \mathcal{B}_M = \{X_1, \dots, X_n\}$$

as well as the set $\mathcal{C}_M$ of connected components of $[-A\,,A] \setminus \mathcal{B}_M$ (sometimes referred to as cells). We also introduce

$$\mathcal{S}_M = \{2t - s : s, t \in \mathcal{T}_M, s < t\}$$

which has to be seen as the set of the possible extinction times of the microscopic fires, recall Lemma II.9.2.

For $\alpha > 0$, we consider the event

$$\Omega_M(\alpha) = \Bigg\{ \min_{\substack{s,t \in \mathcal{T}_M \cup \mathcal{S}_M \\ s \neq t}} |t - s| \geq 2\alpha, \min_{s,t \in \mathcal{T}_M \cup \mathcal{S}_M} |t - (s+1)| \geq 2\alpha,$$
$$\min_{\substack{x,y \in \mathcal{B}_M \cup \{-A,A\}, \\ x \neq y}} |x - y| \geq 2\alpha \Bigg\}$$

which clearly satisfies $\lim_{\alpha \to 0} \mathbb{P}[\Omega_M(\alpha)] = 1$. For any given $\alpha > 0$, there exists $\lambda_\alpha > 0$ such that for all $\lambda \in (0\,, \lambda_\alpha)$, on $\Omega_M(\alpha)$, there holds that

- for all $x, y \in \mathcal{B}_M \cup \{-A, A\}$, with $x \neq y$, $(x)_\lambda \cap (y)_\lambda = \emptyset$;
- the family $\{c_\lambda, c \in \mathcal{C}_M\} \cup \{(x)_\lambda, x \in \mathcal{B}_M\}$ is a partition of I_A^λ.

For $q \in \{1, \dots, n\}$, using the seed processes $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$ and the propagation processes $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$, we build, recall Definition II.4.6, $(\check{\zeta}_t^{\lambda,\pi,q}(i))_{t \geq 0, i \in \mathbb{Z}}$ (the propagation process ignited at (X_q, T_q)), $(i_t^{q,+})_{t \geq 0}$ and $(i_t^{q,-})_{t \geq 0}$ (the corresponding right and left fronts) and $(T_i^q)_{i \in \mathbb{Z}}$ (the associated burning times). We also use $\Omega_{\lambda,\pi}^{P,2A,2A}(X_q, T_q)$, recall Definition II.4.7. We set

$$\Omega_A^{S,P}(\lambda, \pi) = \bigcap_{q=1,\dots,n} \Omega_{\lambda,\pi}^{P,2A,2A}(X_q, T_q).$$

Since π_M is independent of the processes $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$, Lemma II.4.3 implies that $\mathbb{P}\left[\Omega_A^{S,P}(\lambda, \pi)\right]$ tends to 1 when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(0)$.

For $q = 1, \dots, n$, we call $\mathcal{U}_q$ the set of all possible $\mathcal{P} = (\varepsilon; (x_0, t_0), (X_q, T_q), \dots, (x_K, t_K))$ satisfying (PP) where $\{t_0, t_2, \dots, t_K\} \subset \mathcal{T}_M$, $\{x_0, x_2, \dots, x_K\} \subset \mathcal{B}_M$ with $T_q - t_0 > t_2 - T_q$ and with $\varepsilon \in \{-1, 1\}$. For $\mathcal{P} \in \mathcal{U}_q$, we introduce the event $\Omega_\mathcal{P}^{S,P}(\lambda, \pi)$, defined as in Subsection II.9.3, with the Poisson processes $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$. Then we put

$$\Omega_1^{S,P}(\lambda, \pi) = \bigcap_{q=1}^{n} \bigcap_{\mathcal{P} \in \mathcal{U}_q} \Omega_\mathcal{P}^{S,P}(\lambda, \pi),$$

which satisfies $\lim_{\lambda,\pi} \mathbb{P}\left[\Omega_1^{S,P}(\lambda,\pi)\right] = 1$, when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(0)$, thanks to Lemma II.9.3.

We also consider the event $\Omega_2^S(\lambda,\pi)$ on which the following conditions hold: for all $t_1, t_2 \in \mathcal{T}_M$ with $0 < t_2 - t_1 < 1$, for all $q = 1, \dots, n$, there are

$$-\mathbf{m}_\lambda < i_1 < 0 < i_2 < \mathbf{m}_\lambda$$

such that $N^S_{\mathbf{a}_\lambda(t_2+\varkappa_{\lambda,\pi})}(\lfloor \mathbf{n}_\lambda X_q \rfloor + i_j) - N^S_{\mathbf{a}_\lambda t_1}(\lfloor \mathbf{n}_\lambda X_q \rfloor + i_j) = 0$ for $j = 1, 2$. There holds that $\mathbb{P}\left[\Omega_2^S(\lambda,\pi)\right]$ tends to 1 as $\lambda \to$ and $\pi \to \infty$ in the regime $\mathcal{R}(0)$. Indeed, it suffices to prove that almost surely, $\lim_{\substack{\lambda\to 0\\ \pi\to\infty}} \mathbb{P}\left[\Omega_2^S(\lambda,\pi) \,\middle|\, \pi_M\right] = 1$. Since there are a.s. finitely many possibilities for q, t_1, t_2 and since π_M is independent of $(N_t^S(i))_{t\geq 0, i\in\mathbb{Z}}$, it suffices to work with a fixed $q \in \{1, \dots, n\}$ and some fixed $0 < t_2 - t_1 < 1$. The result then follows from Lemma II.9.1-1 together with space/time stationarity.

Next we introduce the event $\Omega_3^S(\lambda,\pi)$ on which the following conditions hold: for all $t_1, t_2 \in \mathcal{T}_M \cup \mathcal{S}_M$,

- if $t_2 - t_1 > 1$, for all $c \in \mathcal{C}_M$, for all $i \in c_\lambda$ with $N^S_{\mathbf{a}_\lambda t_2}(i) - N^S_{\mathbf{a}_\lambda(t_1+\varkappa_{\lambda,\pi})}(i) > 0$;
- if $t_2 - t_1 > 1$, for all $x \in \mathcal{B}_M$, for all $i \in (x)_\lambda$ with $N^S_{\mathbf{a}_\lambda t_2}(i) - N^S_{\mathbf{a}_\lambda(t_1+\varkappa_{\lambda,\pi})}(i) > 0$.

There holds that $\mathbb{P}\left[\Omega_3^S(\lambda,\pi)\right]$ tends to 1 as $\lambda \to$ and $\pi \to \infty$ in the regime $\mathcal{R}(0)$. As previously, it suffices to work with some fixed $t_1, t_2 \in \mathcal{T}_M$, $x \in \mathcal{B}_M$ and $c = (a\,, b) \subset (-A\,, A)$. Observing that $|c_\lambda| \simeq (b-a)\mathbf{n}_\lambda$ and that $|(x)_\lambda| \simeq 2\mathbf{m}_\lambda$, Lemma II.9.1 and space/time stationarity shows the result.

We also need $\Omega_4^{S,P}(\gamma,\lambda,\pi)$, defined for $\gamma > 0$ as follows: for all $q = 1, \dots, n$, for all $\mathcal{M} = ((x_0, t_0), (X_q, T_q))$ such that $t_0 \in \mathcal{T}_M$ with $t_0 < T_q < t_0 + 1$ and $x_0 \in \mathcal{B}_M \setminus \{X_q\}$, there holds that $\left|\Theta^{\lambda,\pi}_{\mathcal{M}} - (T_q - t_0)\right| < \gamma$. Here, $\Theta^{\lambda,\pi}_{\mathcal{M}}$ is defined as in Lemma II.9.2 with the seed processes family $(N_t^S(i))_{t\geq 0, i\in\mathbb{Z}}$ and the propagation processes family $(N_t^P(i))_{t\geq 0, i\in\mathbb{Z}}$. Lemma II.9.2 directly implies that for any $\gamma > 0$, $\mathbb{P}\left[\Omega_4^{S,P}(\gamma,\lambda,\pi)\right]$ tends to 1 as $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(0)$.

We finally introduce the event

$$\Omega(\alpha,\gamma,\lambda,\pi) = \Omega_M(\alpha) \cap \Omega_A^{S,P}(\lambda,\pi) \cap \Omega_1^{S,P}(\lambda,\pi) \cap \Omega_2^S(\lambda,\pi) \cap \Omega_3^S(\lambda,\pi) \cap \Omega_4^{S,P}(\gamma,\lambda,\pi).$$

We have shown that for any $\delta > 0$, there exists $\alpha \in (0\,, 1)$ such that for any $\gamma > 0$, there holds that $\mathbb{P}\left[\Omega(\alpha,\gamma,\lambda,\pi)\right] > 1 - \delta$ for all (λ,π) sufficiently close to the regime $\mathcal{R}(0)$.

II.9.4.3. Heart of the proof

We now handle the main part of the proof.

Consider the $A-$LFFP(0). Observe that by construction, we have, for $c \in \mathcal{C}_M$ and $x, y \in c$, $Z_t(x) = Z_t(y)$ for all $t \in [0\,, T]$, thus we can introduce $Z_t(c)$.

If $x \in \mathcal{B}_M$, it is at the boundary of two cells $c_-, c_+ \in \mathcal{C}_M$ and then we set $Z_t(x_-) = Z_t(c_-)$ and $Z_t(x_+) = Z_t(c_+)$ for all $t \in [0\,, T]$.

If $x \in (-A\,,A) \setminus \mathcal{B}_M$, we put $Z_t(x_-) = Z_t(x_+) = Z_t(x)$ for all $t \in [0\,,T]$.

For $x \in \mathcal{B}_M$ and $t \geq 0$ we set

$$\tilde{H}(x) = \min(H_t(x), 1 - Z_t(x), 1 - Z_t(x_-), 1 - Z_t(x_+)). \tag{II.9.1}$$

Actually $Z_t(x)$ always equals either $Z_t(x_-)$ or $Z_t(x_+)$ and these can be distinct only at a point where has occurred a microscopic fire (that is if $x = X_q$ for some $q \in \{1, \ldots, n\}$ with $T_q < t$ and $Z_{T_q-}(X_q) < 1$).

For all $x \in (-A\,,A)$ and $t \in [0\,,T]$, we put

$$\tau_t(x) = \sup\{s \leq t : Z_s(x_+) = Z_s(x_-) = Z_s(x) = 0\} \in \mathcal{T}_M.$$

For $c \in \mathcal{C}_M$ and $t \in [0\,,T]$, we can define $\tau_t(c)$ as usual with the convention $Z_{0-}(x) = 1$ for all $x \in [-A\,,A]$.

Observe that

$$\text{for } x \notin \mathcal{B}_M,\ Z_t(x) = \min(t - \tau_t(x), 1) \text{ for all } t \in [0\,,T], \tag{II.9.2}$$

$$\text{for } q = 1, \ldots, n,\ Z_t(X_q) = \min(t - \tau_t(X_q), 1) \text{ for all } t \in [0\,,T_q). \tag{II.9.3}$$

We also define, for all $t \in [0\,,T]$, all $i \in I_\lambda^A$,

$$\rho_t^{\lambda,\pi}(i) = \sup\left\{s \leq t : \eta_{\mathbf{a}_\lambda s-}^{\lambda,\pi}(i) = 2\right\} \tag{II.9.4}$$

with the convention $\eta_{0-}^{\lambda,\pi}(i) = 2$ and $\eta_0^{\lambda,\pi}(i) = 0$.

For $t \in [0\,,T]$, consider the event

$$\Omega_t^{\lambda,\pi} = \left\{\forall s \in [0\,,t] \setminus \bigcup_{q=1}^{n} [T_q\,,T_q + \varkappa_{\lambda,\pi}),\ \forall c \in \mathcal{C}_M,\ \forall i \in c_\lambda,\ \left|\rho_s^{\lambda,\pi}(i) - \tau_s(c)\right| \leq \varkappa_{\lambda,\pi}\right\}.$$

Lemma II.9.4. *Let $\alpha > \gamma > 0$. For all $\lambda \in (0\,,\lambda_\alpha)$ and $\pi \geq 1$ sufficiently close to the regime $\mathcal{R}(0)$ in such a way that $\varkappa_{\lambda,\pi} \leq \alpha$, $\Omega_T^{\lambda,\pi}$ a.s. holds on $\Omega(\alpha, \gamma, \lambda, \pi)$.*

Proof. We work on $\Omega(\alpha, \gamma, \lambda, \pi)$ and assume that $\lambda \in (0\,,\lambda_\alpha)$ and $\pi \geq 1$ are such that $\varkappa_{\lambda,\pi} \leq \alpha$. Clearly, $\tau_0(c) = 0$ and $\rho_0^{\lambda,\pi}(i) = 0$ for all $c \in \mathcal{C}_M$ and all $i \in I_A^\lambda$, so that $\Omega_0^{\lambda,\pi}$ a.s. holds. We will show that for $q = 0, \ldots, n-1$, $\Omega_{T_q}^{\lambda,\pi}$ implies $\Omega_{T_{q+1}}^{\lambda,\pi}$. The extension to $\Omega_T^{\lambda,\pi}$ will be straightforward and will be omitted.

We thus fix $q \in \{0, \ldots, n-1\}$ and assume $\Omega_{T_q}^{\lambda,\pi}$. We repeatedly use below that for all $k \leq q$, on the time interval $(T_k\,,T_{k+1})$, there are no fires at all (in $[-A\,,A]$) for the A−LFFP(0) and, on $\Omega_A^{S,P}(\lambda, \pi)$, no burning tree at all (in I_A^λ) during $(\mathbf{a}_\lambda(T_k + \varkappa_{\lambda,\pi})\,, \mathbf{a}_\lambda T_{k+1})$ for the (λ, π, A)−FFP.

Besides, $\eta_{\mathbf{a}_\lambda T_q-}^{\lambda,\pi}(i) = \eta_{\mathbf{a}_\lambda T_q}^{\lambda,\pi}(i)$ for all $i \in I_A^\lambda \setminus \{\lfloor \mathbf{n}_\lambda X_q \rfloor\}$ while

$$\eta_{\mathbf{a}_\lambda T_q}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda X_q \rfloor) = 2\mathbf{1}_{\{\eta_{\mathbf{a}_\lambda T_q-}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda X_q \rfloor) = 1\}}.$$

Step 1. Here we prove that, on $\Omega^{\lambda,\pi}_{T_q}$, for all $1 \le k < q$, if $D_{T_k-}(X_k) = [a\,,b]$, for some $a < b$, $a, b \in \mathcal{B}_M \cup \{-A, A\}$, then

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_k + T^k_{i-\lfloor \mathbf{n}_\lambda X_k \rfloor}}(i) = 2$$

for all $i \in [a\,,b]_\lambda$.

On the one hand, by construction, for all $c \in \mathcal{C}_M$, $c \subset (a\,,b)$, we have $\tau_{T_k}(c) = T_k$. By $\Omega^{\lambda,\pi}_{T_q} \subset \Omega^{\lambda,\pi}_{T_k+\varkappa_{\lambda,\pi}}$, we deduce that $T_k \le \rho^{\lambda,\pi}_{T_k+\varkappa_{\lambda,\pi}}(\lfloor \mathbf{n}_\lambda b \rfloor - \mathbf{m}_\lambda - 1) \le T_k + \varkappa_{\lambda,\pi}$.

On the other hand, recall Lemma II.4.3: on $\Omega^{P,2A,2A}_{\lambda,\pi}(X_k, T_k)$, a burning tree is either a front or has vacant neighbors. Recall that there is no burning tree at all in I^λ_A at time $\mathbf{a}_\lambda T_k-$. Assume for example that there is a site $i \in [\![\lfloor \mathbf{n}_\lambda X_k \rfloor\,, \lfloor \mathbf{n}_\lambda b \rfloor - \mathbf{m}_\lambda - 1]\!]$ such that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_k + T^k_{i-\lfloor \mathbf{n}_\lambda X_k \rfloor}}(i) = 0$. Then the fire starting at $\lfloor \mathbf{n}_\lambda X_k \rfloor$ at time $\mathbf{a}_\lambda T_k$ does not affect the zone $[\![i\,, \lfloor \mathbf{n}_\lambda A \rfloor]\!]$, as seen in **Macro**(0) in Subsection II.4.4. This especially implies that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}(\lfloor \mathbf{n}_\lambda b \rfloor - \mathbf{m}_\lambda - 1) \le 1$ for all $t \in [T_k\,, T_k + \varkappa_{\lambda,\pi}]$ (because no other match falls on I^λ_A during $[\mathbf{a}_\lambda T_k\,, \mathbf{a}_\lambda(T_k + \varkappa_{\lambda,\pi})]$) whence $\rho^{\lambda,\pi}_{T_k+\varkappa_{\lambda,\pi}}(\lfloor \mathbf{n}_\lambda b \rfloor - \mathbf{m}_\lambda - 1) < T_k$, a contradiction.

Step 2. We show that on $\Omega^{\lambda,\pi}_{T_q}$, for all $c \in \mathcal{C}_M$, all $i \in c_\lambda$,

$$\underline{\eta}^{\lambda,\pi}_{\mathbf{a}_\lambda T_q-}(i) \le \eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_q-}(i) \le \overline{\eta}^{\lambda,\pi}_{\mathbf{a}_\lambda T_q-}(i) \tag{II.9.5}$$

where

$$\underline{\eta}^{\lambda,\pi}_{\mathbf{a}_\lambda T_q-}(i) = \min(N^S_{\mathbf{a}_\lambda T_q-}(i) - N^S_{\mathbf{a}_\lambda \tau_{T_q-}(c)+\varkappa_{\lambda,\pi}}(i), 1),$$
$$\overline{\eta}^{\lambda,\pi}_{\mathbf{a}_\lambda T_q-}(i) = \min(N^S_{\mathbf{a}_\lambda T_q-}(i) - N^S_{\mathbf{a}_\lambda \tau_{T_q-}(c)}(i), 1).$$

Indeed, thanks to $\Omega^{S,P}_A(\lambda,\pi) \cap \Omega_M(\alpha)$, there is no burning tree in I^λ_A at time $\mathbf{a}_\lambda T_q-$. Furthermore, for $c \in \mathcal{C}_M$, by $\Omega^{\lambda,\pi}_{T_q}$, we have

$$\tau_{T_q-}(c) \le \rho^{\lambda,\pi}_{T_q-}(i) \le \tau_{T_q-}(c) + \varkappa_{\lambda,\pi} \text{ for all } i \in c_\lambda.$$

By definition, no fire can affect the site i during $(\mathbf{a}_\lambda \rho^{\lambda,\pi}_{T_q-}(i)\,, \mathbf{a}_\lambda T_q)$ whence (II.9.5).

Step 3. We show here that if $Z_{T_q-}(X_q) < 1$, there exist $j_1, j_2 \in (X_q)_\lambda$ such that

$$j_1 < \lfloor \mathbf{n}_\lambda X_q \rfloor < j_2$$
$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}(j_1) = \eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}(j_2) = 0 \text{ for all } t \in [T_q\,, T_q + \varkappa_{\lambda,\pi}].$$

Indeed, since no match falls on X_q during the time interval $[0\,, T_q)$, we have $\tau_{T_q-}(X_q) = T_q - Z_{T_q-}(X_q) = T_k$, for some $0 \le k < q$. Observe that $Z_{T_q-}(X_q) < 1$ implies that $T_q - \tau_{T_q-}(X_q) < 1$.

- If $1 \leq k < q$, then, by construction, we have $X_q \in \mathring{D}_{T_k-}(X_k) = (a\,,b)$, for some $a, b \in \mathcal{B}_M \cup \{-A, A\}$. By $\Omega_M(\alpha)$, we have $|a - X_k| \wedge |b - X_k| > 2\alpha$ whence $(X_q)_\lambda \subset [a\,,b]_\lambda$. We deduce from Step 1 that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_k + T^k_{i-\lfloor \mathbf{n}_\lambda X_k \rfloor}}(i) = 2$ for all $i \in (X_q)_\lambda$. Since we work on $\Omega^S_2(\lambda, \pi)$ and $T_k, T_q \in \mathcal{T}_M$, we know that there are some sites

$$\lfloor \mathbf{n}_\lambda X_k \rfloor - \mathbf{m}_\lambda < j_1 < \lfloor \mathbf{n}_\lambda X_k \rfloor < j_2 < \lfloor \mathbf{n}_\lambda X_k \rfloor + \mathbf{m}_\lambda$$

  such that no seed has fallen on j_1 and j_2 during $[\mathbf{a}_\lambda \tau_{T_q-}(X_q)\,, \mathbf{a}_\lambda(T_q + \varkappa_{\lambda,\pi})]$. Since they are made vacant by the fire k during the time interval $[\mathbf{a}_\lambda T_k\,, \mathbf{a}_\lambda(T_k + \varkappa_{\lambda,\pi}))$, we deduce that they remain vacant during $[\mathbf{a}_\lambda(T_k + \varkappa_{\lambda,\pi})\,, \mathbf{a}_\lambda(T_q + \varkappa_{\lambda,\pi})] \supset [\mathbf{a}_\lambda T_q\,, \mathbf{a}_\lambda(T_q + \varkappa_{\lambda,\pi})]$.

- If $k = 0$, that is if $\tau_{T_q-}(X_q) = 0$ we deduce that $T_q < 1$. We conclude using $\Omega^S_2(\lambda, \pi)$ that there are $j_1 < \lfloor \mathbf{n}_\lambda X_q \rfloor < j_2$ with $j_1, j_2 \in (X_q)_\lambda$ where no seed fall during $[0\,, \mathbf{a}_\lambda(T_q + \varkappa_{\lambda,\pi})]$. Since all the sites are vacant at time 0, we deduce that j_1 and j_2 remain vacant until $\mathbf{a}_\lambda(T_q + \varkappa_{\lambda,\pi})$.

Step 4. Next we check that if $Z_{T_q-}(c) = 1$ for some $c \in \mathcal{C}_M$, then

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_q-}(i) = 1 \text{ for all } i \in c_\lambda.$$

Recalling (II.9.2), we see that $Z_{T_q-}(c) = 1$ implies that $T_q - \tau_{T_q-}(c) \geq 1$ and $T_q - \tau_{T_q-}(c) \geq 1 + 2\alpha$ by $\Omega_M(\alpha)$. Using Step 2, we see that for all $i \in c_\lambda$,

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_q-}(i) \geq \underline{\eta}^{\lambda,\pi}_{\mathbf{a}_\lambda T_q}(i) = \min(N^S_{\mathbf{a}_\lambda T_q-}(i) - N^S_{\mathbf{a}_\lambda \tau_{T_q-}(c)+\varkappa_{\lambda,\pi}}(i), 1).$$

We conclude using $\Omega^S_3(\lambda, \pi)$ that for all $i \in c_\lambda$, $\underline{\eta}^{\lambda,\pi}_{\mathbf{a}_\lambda T_q}(i) = 1$ whence $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_q-}(i) = 1$, as desired.

Step 5. We now prove that if $\tilde{H}_{T_q-}(x) = 0$ for some $x \in \mathcal{B}_M$, then

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_q-}(i) = 1 \text{ for all } i \in (x)_\lambda.$$

Preliminary considerations. Let $k \in \{1, \dots, n\}$ such that $x = X_k$, which is at the boundary of two cells c_-, $c_+ \in \mathcal{C}_M$. We know that $\tilde{H}_{T_q-}(x) = 0$, whence $H_{T_q-}(x) = 0$ and $Z_{T_q-}(x) = Z_{T_q-}(c_+) = Z_{T_q-}(c_-) = 1$. This implies that $T_q \geq 1$ (because $Z_t(x) = t$ for all $t < 1$ and all $x \in [-A\,, A]$) and thus $T_q \geq 1 + 2\alpha$ due to $\Omega_M(\alpha)$.

No fire has concerned $j_g = \lfloor \mathbf{n}_\lambda X_k \rfloor - \mathbf{m}_\lambda - 1 \in (c_-)_\lambda$ during $(\mathbf{a}_\lambda \rho^{\lambda,\pi}_{T_q-}(j_g)\,, \mathbf{a}_\lambda T_q)$. By $\Omega^{\lambda,\pi}_{T_q}$, we deduce that $\tau_{T_q-}(c_-) \leq \rho^{\lambda,\pi}_{T_q-}(j_g) \leq \tau_{T_q-}(c_-) + \varkappa_{\lambda,\pi}$. Recalling (II.9.2), $Z_{T_q-}(c_-) = 1$ implies that $\tau_{T_q-}(c_-) \leq T_q - 1$ whence, by $\Omega_M(\alpha)$, there holds that $\tau_{T_q-}(c_-) < T_q - 1 - 2\alpha$. Using a similar argument for $j_d = \lfloor \mathbf{n}_\lambda X_k \rfloor + \mathbf{m}_\lambda + 1 \in (c_+)_\lambda$, we conclude that no match falling outside $(X_k)_\lambda$ can affect $(X_k)_\lambda$ during $(\mathbf{a}_\lambda(T_q - 1 - \alpha)\,, \mathbf{a}_\lambda T_q)$ (because to affect $(X_k)_\lambda$, a match falling outside $(X_k)_\lambda$ needs to cross j_d or j_g).

Case 1. First assume that $k \geq q$. Then we know that no fire has fallen on $(X_k)_\lambda$ during $[0\,, \mathbf{a}_\lambda T_q)$. Due to the preliminary considerations, we deduce that no fire at all has concerned $(X_k)_\lambda$ during $(\mathbf{a}_\lambda(T_q - 1 - \alpha)\,, \mathbf{a}_\lambda T_q)$. Using $\Omega_3^S(\lambda,\pi)$, we conclude that $(X_k)_\lambda$ is completely occupied at time $\mathbf{a}_\lambda T_q-$.

Case 2. Assume that $k < q$ and $Z_{T_k-}(X_k) = 1$, so that there already has been a macroscopic fire in $(X_k)_\lambda$ (at time $\mathbf{a}_\lambda T_k$). Since $Z_{T_k}(X_k) = 0$ and $Z_{T_q-}(X_k) = 1$, we deduce that $T_q - T_k \geq 1$, whence $T_q - T_k \geq 1 + 2\alpha$ as usual. Since there is no more burning tree in $(X_k)_\lambda$ at time $\mathbf{a}_\lambda(T_k + \varkappa_{\lambda,\pi})$, thanks to $\Omega_{\lambda,\pi}^{P,A}(X_k, T_k)$, we conclude as in Case 1 that no fire at all has concerned $(X_k)_\lambda$ during $(\mathbf{a}_\lambda(T_q - 1 - \alpha)\,, \mathbf{a}_\lambda T_q)$, which implies the claim by $\Omega_3^S(\lambda,\pi)$.

Case 3. Assume that $k < q$ and $Z_{T_k-}(X_k) < 1$ and $T_q - T_k \geq 1$, whence $T_q - T_k \geq 1+2\alpha$ due to $\Omega_M(\alpha)$. Then there already has been a microscopic fire in $(X_k)_\lambda$ (at time $\mathbf{a}_\lambda T_k$). But there are no fire in $(X_k)_\lambda$ during $(\mathbf{a}_\lambda(T_k + \varkappa_{\lambda,\pi})\,, \mathbf{a}_\lambda T_q) \supset (\mathbf{a}_\lambda(T_q - 1 - \alpha)\,, \mathbf{a}_\lambda T_q)$ and we conclude as in Case 2.

Case 4. Assume finally that $k < q$ and $Z_{T_k-}(X_k) < 1$ and $T_q - T_k < 1$, whence $T_q - T_k < 1 - 2\alpha$ due to $\Omega_M(\alpha)$. There has been a microscopic fire in $(X_k)_\lambda$ (at time $\mathbf{a}_\lambda T_k$). Since $H_{T_q-}(X_k) = 0$, we deduce that $T_k + Z_{T_k}(X_k) \leq T_q$, whence $T_k + Z_{T_k}(X_k) \leq T_q - 2\alpha$ by $\Omega_M(\alpha)$. There is $l < k$ such that $\tau_{T_k-}(X_k) = T_l$. We set $\mathcal{M} := ((X_l, T_l), (X_k, T_k))$, recall Subsection II.9.2 (if $l = 0$ *i.e.* $\tau_{T_k-}(X_k) = 0$, set for example $X_0 = 0$).

We first show that

$$(\eta_t^{\lambda,\pi}(i))_{t\in[\mathbf{a}_\lambda T_l, \mathbf{a}_\lambda(T_k+\varkappa_{\lambda,\pi})], i\in(X_k)_\lambda} = (\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t\in[\mathbf{a}_\lambda T_l, \mathbf{a}_\lambda(T_k+\varkappa_{\lambda,\pi})], i\in(X_k)_\lambda}. \tag{II.9.6}$$

Here, the process $(\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t\in[\mathbf{a}_\lambda T_l, \mathbf{a}_\lambda(T_k+\varkappa_{\lambda,\pi})], i\in(X_k)_\lambda}$ is built as in Subsection II.9.2 using the seed processes $(N_t^S(i))_{t\geq 0, i\in\mathbb{Z}}$ and the propagation processes $(N_t^P(i))_{t\geq 0, i\in\mathbb{Z}}$.

- We first assume that $T_l \geq 1$, whence $T_l \geq 1 + 2\alpha$ by $\Omega_M(\alpha)$. Since no match has fallen on $(X_k)_\lambda$ during $[0\,, \mathbf{a}_\lambda T_l]$ and since $Z_{T_l-}(X_k) = 1$, the zone $(X_k)_\lambda$ is completely occupied at time $\mathbf{a}_\lambda T_l-$, recall Case 1. Thus, $(\eta_t^{\lambda,\pi}(i))_{t\geq 0, i\in\mathbb{Z}}$ and $(\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t\geq 0, i\in\mathbb{Z}}$ are equal on $(X_k)_\lambda$ at time $\mathbf{a}_\lambda T_l$. By Step 1, we deduce, that

  $$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_l + T^l_{i-\lfloor \mathbf{n}_\lambda X_l \rfloor}}(i) = 2 \text{ for all } i \in D_{T_l-}(X_l)_\lambda.$$

  Since $(X_k)_\lambda \subset D_{T_l-}(X_l)_\lambda$, we deduce that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_l + T^l_{i-\lfloor \mathbf{n}_\lambda X_l \rfloor}}(i) = 2$ for all $i \in (X_k)_\lambda$. Observe that, with our coupling, the fire l propagates according to the same processes in both cases. Since seeds fall on $(X_k)_\lambda$ according to the same processes and since $(\eta_t^{\lambda,\pi}(i))_{t\geq 0, i\in\mathbb{Z}}$ and $(\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t\geq 0, i\in\mathbb{Z}}$ evolve according to the same rules, we deduce that they remain equals on $(X_k)_\lambda$ during $[\mathbf{a}_\lambda T_l\,, \mathbf{a}_\lambda(T_l + \varkappa_{\lambda,\pi})]$. Next, no fire affects the zone $(X_k)_\lambda$ during $[\mathbf{a}_\lambda(T_l + \varkappa_{\lambda,\pi})\,, \mathbf{a}_\lambda T_k)$ (because to affect the zone $(X_k)_\lambda$, we need $Z_{s-}(c_-) = 1$ or $Z_{s-}(c_+) = 1$ for some $s \in (T_l\,, T_k)$ whereas $Z_s(c_-) = Z_s(c_+) = s - T_l$ for all $s \in [T_l\,, T_k]$) and since seeds fall on $(X_k)_\lambda$ according to the same processes, they are again equal during this time interval. Finally,

$C^P((\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t\geq 0,i\in\mathbb{Z}},(X_k,T_k))\subset (X_k)_\lambda$, recall Lemma II.9.2. We deduce (II.9.6) because the match falling on $\lfloor \mathbf{n}_\lambda X_k \rfloor$ at time $\mathbf{a}_\lambda T_k$ destroys the same zone, since the two processes evolve with the same rules on $(X_k)_\lambda$.

- If $T_l<1$, then by construction $l=0$ and $\tau_{T_k-}(X_k)=0$. We also deduce (II.9.6) using similar arguments as above (this case is easier).

Consider now the zone $C^P=C^P((\eta_t^{\lambda,\pi}(i))_{t\geq 0,i\in\mathbb{Z}},(X_k,T_k))$ destroyed by the match falling on $\lfloor \mathbf{n}_\lambda X_k \rfloor$ at time $\mathbf{a}_\lambda T_k$. This zone is completely occupied at time $\mathbf{a}_\lambda(T_k+\Theta_{\mathcal{M}}^{\lambda,\pi})$: this follows from the definition of $\Theta_{\mathcal{M}}^{\lambda,\pi}$, see Lemma II.9.2, from (II.9.6) and from the preliminary considerations. Using $\Omega_4^S(\gamma,\lambda,\pi)$, we deduce that $T_k+\Theta_{\mathcal{M}}^{\lambda,\pi}\leq T_k+Z_{T_k-}(X_k)+\gamma<T_q$, since $\gamma<\alpha$. Hence C^P is completely occupied at time $\mathbf{a}_\lambda T_q-$.

Consider now $i\in (X_k)_\lambda\backslash C^P$. Then i has not been killed by the fire starting at $\lfloor \mathbf{n}_\lambda X_k \rfloor$. Thus i cannot have been killed during $(\mathbf{a}_\lambda(T_q-1-\alpha),\mathbf{a}_\lambda T_q)$ (due to the preliminary considerations) and we conclude, using $\Omega_3^S(\lambda,\pi)$, that i is occupied at time $\mathbf{a}_\lambda T_q-$. This implies the claim.

Step 6. Let us now prove that if $\tilde H_{T_q-}(x)>0$ and $Z_{T_q-}(x_+)=1$ for some $x\in\mathcal{B}_M$, there is $i_1\in(x)_\lambda$ such that $\eta_{\mathbf{a}_\lambda t}^{\lambda,\pi}(i_1)=0$ for all $t\in[T_q\,,T_q+\varkappa_{\lambda,\pi}]$. Recall that x is at the boundary of two cells c_-,c_+.

We have either $H_{T_q-}(x)>0$ or $Z_{T_q-}(c_-)<1$ (because $Z_{T_q-}(c_+)=1$ by assumption). Clearly, $x=X_k$ for some $k<q$, with $Z_{T_k-}(X_k)<1$ (else, we would have $H_t(x)=0$ and $Z_t(c_-)=Z_t(c_+)$ for all $t\in[0\,,T_q)$). Thus, recalling (II.9.2), $T_k-Z_{T_k-}(X_k)=\tau_{T_k-}(X_k)=T_l$, for some $l<k$.

As checked in case 4 in the previous Step, on $\Omega(\alpha,\gamma,\lambda,\pi)$, setting $\mathcal{M}=((X_l,T_l),(X_k,T_k))$ (if $l=0$, set for example $X_0=0$)

$$(\eta_t^{\lambda,\pi}(i))_{t\in[\mathbf{a}_\lambda T_l,\mathbf{a}_\lambda(T_k+\varkappa_{\lambda,\pi})],i\in(X_k)_\lambda}=(\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t\in[\mathbf{a}_\lambda T_l,\mathbf{a}_\lambda(T_k+\varkappa_{\lambda,\pi})],i\in(X_k)_\lambda}$$

where the process $(\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t\in[\mathbf{a}_\lambda T_l,\mathbf{a}_\lambda(T_k+\varkappa_{\lambda,\pi})],i\in(X_k)_\lambda}$ is built as in Subsection II.9.2 using the seed processes $(N_t^S(i))_{t\geq 0,i\in\mathbb{Z}}$ and the propagation processes $(N_t^P(i))_{t\geq 0,i\in\mathbb{Z}}$. Hence, either $l=0$ whence $\eta_0^{\lambda,\pi}(i)=0$ for all $i\in(X_k)_\lambda$ or all the sites in $(X_k)_\lambda$ burn at least on time during $[\mathbf{a}_\lambda T_l\,,\mathbf{a}_\lambda(T_l+\varkappa_{\lambda,\pi}))$.

Case 1. Assume first that $H_{T_q-}(x)>0$. Then by construction, there holds $T_k+Z_{T_k-}(X_k)>T_q>T_k$, whence by $\Omega_M(\alpha)$, $T_k+Z_{T_k-}(X_k)>T_q+2\alpha>T_k+4\alpha$.

Consider $C^P=C^P((\eta_t^{\lambda,\pi}(i))_{t\geq 0,i\in\mathbb{Z}},(X_k,T_k))$ the zone destroyed by the match falling on $\lfloor \mathbf{n}_\lambda X_k \rfloor$ at time $\mathbf{a}_\lambda T_k$. By $\Omega_2^S(\lambda,\pi)$ and (II.9.6), we have $C^P\subset(X_k)_\lambda$ (because $T_k-Z_{T_k-}(X_k)$ and T_k belong to $\mathcal{T}_M$, because $0<Z_{T_k-}(X_k)<1$ and because all the sites in $(X_k)_\lambda$ have been made vacant during $[\mathbf{a}_\lambda T_l\,,\mathbf{a}_\lambda(T_l+\varkappa_{\lambda,\pi})))$.

By Definition of $\Theta_{\mathcal{M}}^{\lambda,\pi}$, see Lemma II.9.2 and by (II.9.6), we deduce that C^P is not completely occupied at time $\mathbf{a}_\lambda(T_k+\Theta_{\mathcal{M}}^{\lambda,\pi})$ (because in both cases, seeds fall on $(X_k)_\lambda$ according to the same processes). But by $\Omega_4^{S,P}(\gamma,\lambda,\pi)$, we see that $\Theta_{\mathcal{M}}^{\lambda,\pi}\geq Z_{T_k-}(X_k)-\gamma$, whence $T_k + \Theta^{\lambda,\pi}_{\mathcal{M}} \geq T_k + Z_{T_k-}(X_k) - \gamma + 2\alpha > T_q + \varkappa_{\lambda,\pi}$ since $\gamma < \alpha$ and $\varkappa_{\lambda,\pi} < \alpha$. All this implies that there is a vacant site in C^P during $[\mathbf{a}_\lambda T_q \,, \mathbf{a}_\lambda(T_q + \varkappa_{\lambda,\pi})]$.

Case 2. Assume next that $H_{T_q-}(x) = 0$ and that $T_q - T_l < 1$ (whence $T_q - T_l < 1 - 2\alpha$).

- If $l \geq 1$, recall that a match has fallen (in the limit process) on $X_l \in \mathcal{B}_M$ at time $T_l \in \mathcal{T}_M$ with $X_k \in \mathring{D}_{T_l-}(X_l)$. Since T_l and T_q belong to $\mathcal{T}_M$ and since their difference is smaller than 1 by assumption, $\Omega_2^S(\lambda, \pi)$ guarantees us the existence of $i_1 \in (X_k)_\lambda$, such that no seed fall on i_1 during $[\mathbf{a}_\lambda T_l \,, \mathbf{a}_\lambda(T_q + \varkappa_{\lambda,\pi})]$. Since all the sites in $(X_k)_\lambda$ have been made vacant during the time interval $[\mathbf{a}_\lambda T_l \,, \mathbf{a}_\lambda(T_l + \varkappa_{\lambda,\pi})]$ (see Step 1), one easily concludes that i_1 is vacant during $[\mathbf{a}_\lambda T_q \,, \mathbf{a}_\lambda(T_q + \varkappa_{\lambda,\pi})]$.

- If $l = 0$ that is if $0 < T_q < 1$, there holds $0 < T_q < 1 - 2\alpha$ by $\Omega_M(\alpha)$. We conclude using $\Omega_2^S(\lambda, \pi)$ that there is a site $i_1 \in (X_k)_\lambda$ where no seed has fallen during $[0 \,, \mathbf{a}_\lambda(T_q + \varkappa_{\lambda,\pi})]$ whence $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda s}(i_1) = 0$ for all $s \in [\mathbf{a}_\lambda T_q \,, \mathbf{a}_\lambda(T_q + \varkappa_{\lambda,\pi})]$, as desired.

Case 3. Assume finally that $H_{T_q-}(x) = 0$ and that $T_q - [T_k - Z_{T_k-}(X_k)] \geq 1$, whence $T_q - [T_k - Z_{T_k-}(X_k)] \geq 1 + 2\alpha$ by $\Omega_M(\alpha)$. Since $H_{T_q-}(x) = 0$, there holds $Z_{T_q-}(c_-) < 1 = Z_{T_q-}(c_+)$ and $T_k + Z_{T_k-}(X_k) \leq T_q$, so that $T_k + Z_{T_k-}(X_k) \leq T_q - 2\alpha$.

We aim to use the event $\Omega_1^{S,P}(\lambda, \pi)$. We introduce

$$t_0 = T_k - Z_{T_k-}(X_k) = \tau_{T_k-}(X_k) = T_l.$$

Observe that $\tau_{T_k-}(c_-) = \tau_{T_k-}(c_+) = \tau_{T_k-}(x)$ because there has been no fire (exactly) at x during $[0 \,, T_k)$. Thus $Z_{t_0-}(x) = Z_{t_0-}(x_-) = Z_{t_0-}(x_+) = 1$ and $Z_{t_0}(x) = Z_{t_0}(c_-) = Z_{t_0}(c_+) = 0$ (using the convention $Z_{0-}(y) = 1$ for all $y \in [-A \,, A]$).

Set now $t_1 = T_k$. Observe that $0 < t_1 - t_0 < 1$. Necessarily, $Z_t(c_-)$ has jumped to 0 at least one time between t_0 and T_q- (else, one would have $Z_{T_q-}(c_-) = 1$, since $T_q - t_0 \geq 1$ by assumption) and this jump occurs after $t_0 + 1 > t_1$ (since a jump of $Z_t(c_-)$ requires that $Z_t(c_-) = 1$, and since for all $t \in [t_0 \,, t_0 + 1)$, $Z_t(c_-) = t - t_0 < 1$).

We thus may denote by $t_2 < t_3 < \cdots < t_K$, for some $K \geq 2$, the successive times of jumps of the process $(Z_t(c_-), Z_t(c_+))$ during $(t_0 + 1 \,, T_q)$ and say $x_2, \ldots, x_K$ the corresponding locations of the fires. We also put $\varepsilon = 1$ if t_2 is a jump of $Z_t(c_+)$ and $\varepsilon = -1$ else.

Then we observe that $Z_t(c_-)$ and $Z_t(c_+)$ do never jump to 0 at the same time during $(t_0 \,, T_q)$ (else, it would mean that they are killed by the same fire at some time u, whence necessarily, $H_r(u) = 0$ and $Z_r(c_-) = Z_r(c_+)$ for all $r \in (u \,, T_q)$). Furthermore, there is always at least one jump of $(Z_t(c_-), Z_t(c_+))$ in any time interval of length 1 (during $[t_0 + 1 \,, T_q)$), because else, $Z_t(c_+)$ and $Z_t(c_-)$ would both become equal to 1 and thus would remain equal forever. Finally, observe that two jumps of $Z_t(c_-)$ cannot occur in a time interval of length 1 (since a jump of $Z_t(c_-)$ requires that $Z_t(c_-) = 1$) and the same thing holds for $Z_t(c_+)$.

Consequently, the family $\mathcal{P} = \{\varepsilon; (x_0, t_0), (X_k, T_k), \ldots, (x_K, t_K)\}$ necessarily satisfies the condition (PP) of Subsection II.9.3.

Next, there holds that $t_2 - t_1 < Z_{T_k-}(X_k) = t_1 - t_0$, because else, we would have $H_{t_2-}(X_k) = 0$ and thus the fire destroying c_+ (or c_-) at time t_2 would also destroy c_- (or c_+), we thus would have $Z_{t_2}(c_+) = Zt_2(c_-) = 0$, so that $Z_t(c_+)$ and $Z_t(c_-)$ would remain equal forever. Furthermore, we have $t_K < T_q < t_K + 1$ because else, we would have $Z_{T_q-}(c_+) = Z_{T_q-}(c_-) = 1$.

Finally, we check that

$$(\eta_t^{\lambda,\pi}(i))_{t\in[\mathbf{a}_\lambda t_0,\mathbf{a}_\lambda(t_K+\varkappa_{\lambda,\pi})],i\in(X_k)_\lambda} = (\zeta_t^{\lambda,\pi,\mathcal{P}}(i))_{t\in[\mathbf{a}_\lambda t_0,\mathbf{a}_\lambda(t_K+\varkappa_{\lambda,\pi})],i\in(X_k)_\lambda},$$

this last process being built upon the families $(N_t^S(i))_{t\geq 0,i\in\mathbb{Z}}$ and $(N_t^P(i))_{t\geq 0,i\in\mathbb{Z}}$ as in Subsection II.9.2. Indeed, seeds fall according to the same processes and fires propagate according to the same processes on $(X_k)_\lambda$. We already have checked that $(\eta_t^{\lambda,\pi}(i))_{t\geq 0,i\in\mathbb{Z}}$ and $(\zeta_t^{\lambda,\pi,\mathcal{P}}(i))_{t\geq 0,i\in\mathbb{Z}}$ are equal on $(X_k)_\lambda$ during the time interval $[\mathbf{a}_\lambda t_0\,,\mathbf{a}_\lambda(T_k+\varkappa_{\lambda,\pi})]$. Nothing happens on $(X_k)_\lambda$ during $[\mathbf{a}_\lambda(T_k+\varkappa_{\lambda,\pi})\,,\mathbf{a}_\lambda t_2)$. In both cases (say $\varepsilon=-1$), a match falls on $\lfloor \mathbf{n}_\lambda x_2\rfloor \in [\![-\lfloor \mathbf{n}_\lambda A\rfloor\,, \lfloor \mathbf{n}_\lambda X_k\rfloor - \mathbf{m}_\lambda]\!]$ at time $\mathbf{a}_\lambda t_2$. This fire destroys destroys the zone containing $\lfloor \mathbf{n}_\lambda X_k\rfloor - \mathbf{m}_\lambda$ (by definition of $\zeta^{\lambda,\pi,\mathcal{P}}$ and because, by construction, $D_{t_2-}(x_2) = [a\,, X_k]$, for some $a \in \mathcal{B}_M \cup \{-A\}$, whence $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_2-}(j) = 1$ for all $j \in [\![\lfloor \mathbf{n}_\lambda x_2\rfloor\,, \lfloor \mathbf{n}_\lambda X_m\rfloor - \mathbf{m}_\lambda]\!]$, see Steps 4 and 5 above) at the same time, since with our coupling, the second fire spreads according to the same rules and to the same processes in both cases. This implies that $(\eta_t^{\lambda,\pi}(i))_{t\geq 0,i\in\mathbb{Z}}$ and $(\zeta_t^{\lambda,\pi,\mathcal{P}}(i))_{t\geq 0,i\in\mathbb{Z}}$ are also equal on $(X_k)_\lambda$ during the time interval $[\mathbf{a}_\lambda(T_k+\varkappa_{\lambda,\pi})\,,\mathbf{a}_\lambda(t_2+\varkappa_{\lambda,\pi})]$. And so on.

We thus can use $\Omega_1^{S,P}(\lambda,\pi)$ and conclude that there is a site i_1 in $(X_k)_\lambda$ which is vacant during $[\mathbf{a}_\lambda(t_K+\varkappa_{\lambda,\pi})\,,\mathbf{a}_\lambda(t_K+1)]$ for $(\zeta_t^{\lambda,\pi,\mathcal{P}}(i))_{t\geq 0,i\in\mathbb{Z}}$. Since seeds fall on $(X_k)_\lambda$ according to the same processes, we deduce that there is also a vacant site in $(X_k)_\lambda$ during $[\mathbf{a}_\lambda(t_K+\varkappa_{\lambda,\pi})\,,\mathbf{a}_\lambda(t_K+1)] \subset [\mathbf{a}_\lambda T_q\,,\mathbf{a}_\lambda(T_q+\varkappa_{\lambda,\pi})]$ for the $(\lambda,\pi,A)-$FFP, as desired.

Step 7. We now conclude. We put $z := Z_{T_q-}(X_q)$ and consider separately the cases $z\in(0\,,1)$ and $z=1$. Observe that $z=0$ do never happens, since by construction, $Z_{T_q-}(X_q) = \min(Z_{T_{q-1}}(X_q) + T_q - T_{q-1}, 1) > 0$ and since $T_q > T_{q-1}$.

Case $z\in(0\,,1)$. Then in the $A-$LFFP(0), we have $Z_{T_q-}(X_q) = Z_{T_q}(X_q)$ for all $x\in(-A\,,A)$ whence $\tau_{T_q-}(c) = \tau_{T_q}(c) = \tau_{T_q+\varkappa_{\lambda,\pi}}(c)$ for all $c\in\mathcal{C}_M$. Using Step 3, as seen in **Micro**(0) in Subsection II.4.4, we see that the match falling on $\lfloor \mathbf{n}_\lambda X_q\rfloor$ at time $\mathbf{a}_\lambda T_q$ destroys nothing outside $[\![j_1\,,j_2]\!] \subset (X_q)_\lambda$ and there is no more burning tree in I_A^λ at time $\mathbf{a}_\lambda(T_q+\varkappa_{\lambda,\pi})$. We deduce that $\rho_s^{\lambda,\pi}(i) = \rho_{T_q}^{\lambda,\pi}(i)$ for all $s\in[T_q\,,T_q+\varkappa_{\lambda,\pi}]$ and all $i\notin(X_q)_\lambda$. Thus, applying $\Omega_{T_q}^{\lambda,\pi}$, we deduce that for all $c\in\mathcal{C}_M$ and all $i\in c_\lambda$,

$$\tau_{T_q+\varkappa_{\lambda,\pi}}(c) = \tau_{T_q}(c) \leq \rho_{T_q}^{\lambda,\pi}(i) = \rho_{T_q+\varkappa_{\lambda,\pi}}^{\lambda,\pi}(i) \leq \tau_{T_q}(c) + \varkappa_{\lambda,\pi} = \tau_{T_q+\varkappa_{\lambda,\pi}}(c) + \varkappa_{\lambda,\pi}.$$

Thus, on $\Omega(\alpha,\gamma,\lambda,\pi)$, $\Omega_{T_q}^{\lambda,\pi}$ implies $\Omega_{T_q+\varkappa_{\lambda,\pi}}^{\lambda,\pi}$. Since no match falls on I_A^λ during $(\mathbf{a}_\lambda(T_q+\varkappa_{\lambda,\pi})\,,\mathbf{a}_\lambda T_{q+1})$ and since $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_{q+1}-}(i) = \eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_{q+1}}(i)$ for all $i\neq\lfloor \mathbf{n}_\lambda X_{q+1}\rfloor$, we deduce that on $\Omega(\alpha,\gamma,\lambda,\pi)$, for all $c\in\mathcal{C}_M$ and all $i\in c_\lambda$,

$$\rho_{T_q+\varkappa_{\lambda,\pi}}^{\lambda,\pi}(i) = \rho_{T_{q+1}}^{\lambda,\pi}(i) \text{ and } \tau_{T_q+\varkappa_{\lambda,\pi}}(c) = \tau_{T_{q+1}}(c).$$

All this implies that on $\Omega(\alpha, \gamma, \lambda, \pi)$, $\Omega^{\lambda,\pi}_{T_q}$ implies $\Omega^{\lambda,\pi}_{T_{q+1}}$ when $z \in (0\,,1)$.

Case $z = 1$. Then there are $a, b \in \mathcal{B}_M \cup \{-A, A\}$ such that $D_{T_q-}(X_q) = [a\,,b]$. We assume that $a, b \in \mathcal{B}_M$, the other cases being treated similarly. By construction, we know that for all $c \in \mathcal{C}_M$ with $c \subset (a\,,b)$, $Z_{T_q-}(c) = 1$, for all $x \in \mathcal{B}_M \cap (a\,,b)$, $\tilde{H}_{T_q-}(x) = 0$ while finally $\tilde{H}_{T_q-}(a) > 0$ and $\tilde{H}_{T_q-}(b) > 0$.

For the A−LFFP(0), we have

(i) $\tau_{T_q}(c) = T_q$ for all $c \in \mathcal{C}_M$ with $c \subset (a\,,b)$,

(ii) $\tau_{T_q}(c) = \tau_{T_q-}(c)$ for all $c \in \mathcal{C}_M$ with $c \cap (a\,,b) = \emptyset$.

Next, using Steps 4, 5, using Step 6 for a (and a very similar result for b), we immediately check that the fire occurring on $\lfloor \mathbf{n}_\lambda X_q \rfloor$ at time $\mathbf{a}_\lambda T_q$, as seen in **Macro**(0) in Subsection II.4.4,

- destroys completely all the cells $c \in \mathcal{C}_M$ with $c \subset (a\,,b)$,
- destroys completely all the zones $(x)_\lambda$ with $x \in \mathcal{B}_M \cap (a\,,b)$,
- does not destroy completely $(a)_\lambda$ nor $(b)_\lambda$,
- does not destroy at all the sites $i \in I^\lambda_A$ with $i \notin [\![\lfloor \mathbf{n}_\lambda a \rfloor - \mathbf{m}_\lambda\,, \lfloor \mathbf{n}_\lambda b \rfloor + \mathbf{m}_\lambda]\!]$.

Consequently, we have, for all $c \in \mathcal{C}_M$ with $c \subset (a\,,b)$ and all $i \in (c)_\lambda$,

$$\tau_{T_q+\varkappa_{\lambda,\pi}}(c) = \tau_{T_q}(c) = T_q \le \rho^{\lambda,\pi}_{T_q+\varkappa_{\lambda,\pi}}(i) \le T_q + \varkappa_{\lambda,\pi} = \tau_{T_q}(c) + \varkappa_{\lambda,\pi} = \tau_{T_q+\varkappa_{\lambda,\pi}}(c) + \varkappa_{\lambda,\pi},$$

while if $c \cap (a\,,b) = \emptyset$, for all $i \in (c)_\lambda$,

$$\begin{aligned}\tau_{T_q+\varkappa_{\lambda,\pi}}(c) = \tau_{T_q}(c) = \tau_{T_q-}(c) \le \rho^{\lambda,\pi}_{T_q-}(i) = \rho^{\lambda,\pi}_{T_q+\varkappa_{\lambda,\pi}}(i) \\ \le \tau_{T_q-}(c) + \varkappa_{\lambda,\pi} = \tau_{T_q}(c) + \varkappa_{\lambda,\pi} = \tau_{T_q+\varkappa_{\lambda,\pi}}(c) + \varkappa_{\lambda,\pi}.\end{aligned}$$

We conclude that when $z = 1$, $\Omega^{\lambda,\pi}_{T_q}$ implies $\Omega^{\lambda,\pi}_{T_q+\varkappa_{\lambda,\pi}}$. Since no match falls on I^λ_A during $[\mathbf{a}_\lambda(T_q + \varkappa_{\lambda,\pi})\,, \mathbf{a}_\lambda T_{q+1})$ and since $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_{q+1}-}(i) = \eta^{\lambda,\pi}_{\mathbf{a}_\lambda T_{q+1}}(i)$ for all $i \neq \lfloor \mathbf{n}_\lambda X_{q+1} \rfloor$, we deduce that on $\Omega(\alpha, \gamma, \lambda, \pi)$, $\Omega^{\lambda,\pi}_{T_q+\varkappa_{\lambda,\pi}}$ implies $\Omega^{\lambda,\pi}_{T_{q+1}}$.

All this implies that on $\Omega(\alpha, \gamma, \lambda, \pi)$, $\Omega^{\lambda,\pi}_{T_q}$ implies $\Omega^{\lambda,\pi}_{T_{q+1}}$ when $z = 1$. This completes the proof. □

II.9.5. Proof of Theorem II.6.1 for $p = 0$

We finally give the proof of the Theorem II.6.1 in the case $p = 0$. The proof is closely related to the proof in the case $p > 0$, recall Subsection II.8.5.

Proof. Let us fix $x_0 \in (-A\,, A)$, $t_0 \in (0\,, T)$ and $\varepsilon > 0$. We will prove that with our coupling (see Subsection II.9.4.1), when $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(0)$, there holds that

(a) $\lim_{\lambda,\pi} \mathbb{P}\left[\boldsymbol{\delta}(D^{\lambda,\pi}_{t_0}(x_0), D_{t_0}(x_0)) > \varepsilon\right] = 0$;

(b) $\lim_{\lambda,\pi} \mathbb{P}\left[\boldsymbol{\delta}_T(D^{\lambda,\pi}(x_0), D(x_0)) > \varepsilon\right] = 0$;

(c) $\lim_{\lambda,\pi} \mathbb{P}\left[\left|Z^{\lambda,\pi}_t(x_0) - Z_t(x_0)\right| > \varepsilon\right] = 0$;

(d) $\lim_{\lambda,\pi} \mathbb{P}\left[\int_0^T \left|Z^{\lambda,\pi}_t(x_0) - Z_t(x_0)\right| \mathrm{d}t > \varepsilon\right] = 0$;

(e) $\lim_{\lambda,\pi} \mathbb{P}\left[\left|W^{\lambda,\pi}_{t_0}(x_0) - Z_{t_0}(x_0)\right| > \varepsilon\right] = 0$, where

$$W^{\lambda,\pi}_{t_0}(x_0) = \left(\frac{\log(|C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}, \lfloor \mathbf{n}_\lambda x_0 \rfloor)|)}{\log(1/\lambda)} \mathbf{1}_{\{|C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}, \lfloor \mathbf{n}_\lambda x_0 \rfloor)| \geq 1\}}\right) \wedge 1.$$

These points will clearly imply the result.
First, we introduce the event $\Omega^{x_0,t_0}_{A,T}(\alpha,\lambda,\pi)$ on which

(i) $x_0 \notin \cup_{y \in \mathcal{B}_M} (y - 2\alpha\,, y + 2\alpha)$;

(ii) for all $s \in \mathcal{T}_M \cup \mathcal{S}_M$ with $s \leq t_0$, there holds that $t_0 - s > 2\alpha$;

(iii) if $t_0 \neq 1$, for all $s \in \mathcal{T}_M \cup \mathcal{S}_M$ with $s \leq t_0$, there holds that $|t_0 - (s+1)| > 2\alpha$;

(iv) if $t_0 > 1$, for all $i \in I^\lambda_A$, $N^S_{\mathbf{a}_\lambda t_0}(i) - N^S_{\mathbf{a}_\lambda (t_0 - 1)}(i) > 0$;

(v) if $t_c = t_0 - \tau_{t_0-}(x_0) < 1$, there are i_1 and i_2 such that

$$-\lfloor \lambda^{-(t_c+\alpha)} \rfloor < i_1 < -\lfloor \lambda^{-(t_c-\alpha)} \rfloor < 0 < \lfloor \lambda^{-(t_c-\alpha)} \rfloor < i_2 < \lfloor \lambda^{-(t_c+\alpha)} \rfloor$$

and such that

- $N^S_{\mathbf{a}_\lambda t_0}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1) - N^S_{\mathbf{a}_\lambda \tau_{t_0-}(x_0)}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1) = 0$ whereas $N^S_{\mathbf{a}_\lambda t_0}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2) - N^S_{\mathbf{a}_\lambda \tau_{t_0-}(x_0)}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2) = 0$;
- for all $j \in [\![-\lfloor \lambda^{-(t_c-\alpha)} \rfloor, \lfloor \lambda^{-(t_c-\alpha)} \rfloor]\!]$,

$$N^S_{\mathbf{a}_\lambda t_0}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + j) - N^S_{\mathbf{a}_\lambda (\tau_{t_0-}(x_0) + \varkappa_{\lambda,\pi})}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + j) > 0.$$

Since $t_0 - \tau_{t_0-}(x_0) = 1$ occurs with positive probability only if $t_0 = 1$ (and $\tau_{t_0-}(x_0) = 0$), the probability of the three first points clearly tend to 1 when α tends to 0. Since $(\tau_t(x_0))_{t\geq 0}$ is independent of $(N^S_t(i))_{t\geq 0, i\in\mathbb{Z}}$ and since $(\tau_t(x_0))_{t\geq 0} \subset \mathcal{T}_M \cup \mathcal{S}_M$, the probability of the two last points also tend to 1 as $\alpha \to 0$ and $\lambda \to 0$ and $\pi \to \infty$ in the regime $\mathcal{R}(0)$, thanks to Lemma II.9.1-4,6,7 and space/time stationarity (recall that $\varkappa_{\lambda,\pi} \to 0$). All this implies that for all $\delta > 0$, there is $\alpha > 0$ such that $\mathbb{P}\left[\Omega^{x_0,t_0}_{A,T}(\alpha,\lambda,\pi)\right] > 1 - \delta$ for all (λ,π) sufficiently close to the regime $\mathcal{R}(0)$.

Let us now fix $\delta > 0$. We consider $\alpha_0 \in (0\,,\varepsilon)$, $\gamma_0 \in (0\,,\alpha_0)$, $\lambda_0 \in (0\,,1)$ and $\epsilon_0 \in (0\,,1)$ such that for all $\lambda \in (0\,,\lambda_0)$ and all $\pi \geq 1$ in such a way that $\mathbf{n}_\lambda/(\mathbf{a}_\lambda \pi) < \epsilon_0$, we have

$$\mathbb{P}\left[\Omega(\alpha_0, \gamma_0, \lambda, \pi) \cap \Omega^{x_0,t_0}_{A,T}(\alpha_0, \lambda, \pi)\right] > 1 - \delta.$$

We then consider $\lambda_1 \in (0\,,\lambda_0)$ and $\epsilon_1 \in (0\,,\epsilon_0)$ such that for all all $\lambda \in (0\,,\lambda_1)$ and all $\pi \geq 1$ in such a way that $\mathbf{n}_\lambda/(\mathbf{a}_\lambda \pi) < \epsilon_1$, we have

- $\varkappa_{\lambda,\pi} \leq \alpha_0$;
- $\alpha_0 + \log(\mathbf{a}_\lambda)/\log(1/\lambda) < \varepsilon$;
- $4\mathbf{m}_\lambda/\mathbf{n}_\lambda \leq \varepsilon$;
- $1/(2\mathbf{m}_\lambda \lambda^{t_c - 2\varepsilon}) \leq \delta$ and $1/(2\mathbf{m}_\lambda \lambda^{t_c + \varkappa_{\lambda,\pi}}) \leq \delta$ if $t_c < 1$.

All this can be done properly by using the fact that $\varkappa_{\lambda,\pi} \to 0$ and $\mathbf{m}_\lambda/\mathbf{n}_\lambda \to 0$.

In the rest of the proof, we consider $\lambda \in (0\,,\lambda_1)$ and $\pi \geq 1$ in such a way that $\mathbf{n}_\lambda/(\mathbf{a}_\lambda \pi) \leq \epsilon_1$. Observe that, on $\Omega^{x_0,t_0}_{A,T}(\alpha_0, \lambda, \pi)$, we have $\tau_{t_0-}(x_0) = \tau_{t_0}(x_0)$ and $(x_0)_\lambda \cap \left(\bigcup_{x \in \mathcal{B}_M} (x)_\lambda\right) = \emptyset$. We call $c_0 \in \mathcal{C}_M$ the cell containing x_0.

Step 1. As in Subsection II.8.5, Steps 1 and 2, (a) (which holds for an arbitrary value of $t_0 \in (0\,,T)$) implies (b) and (c) implies (d).

Step 2. Due to Lemma II.9.4, we know that, on $\Omega(\alpha_0, \gamma_0, \lambda, \pi) \cap \Omega^{x_0,t_0}_{A,T}(\alpha_0, \lambda, \pi)$, since $t_0 > \tau_{t_0}(x_0) + 3\alpha_0$, for all $i \in (x_0)_\lambda$,

$$\tau_{t_0}(c_0) \leq \rho^{\lambda,\pi}_{t_0}(i) \leq \tau_{t_0}(c_0) + \varkappa_{\lambda,\pi}.$$

For all $i \in (x_0)_\lambda$, since $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) \leq 1$, there holds

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) = \min(N^{S,\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) - N^{S,\lambda,\pi}_{\mathbf{a}_\lambda \rho^{\lambda,\pi}_{t_0}(i)}(i), 1).$$

Thus, for all $i \in (x_0)_\lambda$,

$$\underline{\eta}^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) \leq \eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) \leq \overline{\eta}^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i)$$

where

$$\underline{\eta}^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) := \min(N^S_{\mathbf{a}_\lambda t_0}(i) - N^S_{\mathbf{a}_\lambda(\tau_{t_0}(x_0) + \varkappa_{\lambda,\pi})}(i), 1),$$
$$\overline{\eta}^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) := \min(N^S_{\mathbf{a}_\lambda t_0}(i) - N^S_{\mathbf{a}_\lambda \tau_{t_0}(x_0)}(i), 1).$$

We also recall that by construction, $(\tau_t(x_0))_{t\geq 0}$ is independent of $(N^S_t(i))_{t\geq 0, i\in\mathbb{Z}}$.

Step 3. Here we prove (e). We work on $\Omega(\alpha_0,\gamma_0,\lambda,\pi)\cap\Omega_{A,T}^{x_0,t_0}(\alpha_0,\lambda,\pi)$. By Step 2 and point (v) of the event $\Omega_{A,T}^{x_0,t_0}(\alpha_0,\lambda,\pi)$, we observe that if $0<t_c=t_0-\tau_{t_0}(x_0)<1$, then

$$\begin{aligned}
&[\![\lfloor \mathbf{n}_\lambda x_0\rfloor - \lfloor \lambda^{-(t_c-\alpha_0)}\rfloor, \lfloor \mathbf{n}_\lambda x_0\rfloor + \lfloor \lambda^{-(t_c-\alpha_0)}\rfloor]\!]\\
&\qquad\subset C(\underline{\eta}^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}, \lfloor \mathbf{n}_\lambda x_0\rfloor) \subset C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}, \lfloor \mathbf{n}_\lambda x_0\rfloor) \subset C(\overline{\eta}^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}, \lfloor \mathbf{n}_\lambda x_0\rfloor)\\
&\qquad\qquad\subset [\![\lfloor \mathbf{n}_\lambda x_0\rfloor - \lfloor \lambda^{-(t_c+\alpha)}\rfloor, \lfloor \mathbf{n}_\lambda x_0\rfloor + \lfloor \lambda^{-(t_c+\alpha)}\rfloor]\!].
\end{aligned}$$

Thus, this implies that

$$|W^{\lambda,\pi}_{t_0}(x_0)-(t_0-\tau_{t_0}(x_0))| \le \alpha_0 + \frac{\log(2)}{\log(1/\lambda)} < \varepsilon.$$

If now $t_0-\tau_{t_0}(x_0)>1$, then $t_0-\tau_{t_0}(x_0)>1+2\alpha_0$ thanks to $\Omega_{A,T}^{x_0,t_0}(\alpha_0,\lambda,\pi)$. Then Step 2 and point (iv) of $\Omega_{A,T}^{x_0,t_0}(\alpha_0,\lambda,\pi)$ imply that $(x_0)_\lambda\subset C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0},\lfloor \mathbf{n}_\lambda x_0\rfloor)$ whence $|C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0},\lfloor \mathbf{n}_\lambda x_0\rfloor)|\ge 2\mathbf{m}_\lambda$. Consequently,

$$W^{\lambda,\pi}_{t_0}(x_0) \ge 1-\frac{\log(\mathbf{a}_\lambda)}{\log(1/\lambda)} > 1-\varepsilon.$$

It only remains to study what happens when $t_0=1$. By construction, we have $\tau_{t_0}(x_0)=0$. Observe that on $\Omega(\alpha,\gamma,\lambda,\pi)$, a match falling on $\lfloor \mathbf{n}_\lambda X_k\rfloor$ at time $\mathbf{a}_\lambda T_k\le 1$, for some $k\in\{1,\dots,n\}$, does not affect the zone outside $(X_k)_\lambda$. Thus, for all $i\in(x_0)_\lambda$,

$$\eta^{\lambda,\pi}_{\mathbf{a}_\lambda}(i)=\min(N^S_{\mathbf{a}_\lambda}(i),1).$$

Using point (iv) of the event $\Omega_{A,T}^{x_0,t_0}(\alpha_0,\lambda,\pi)$, we deduce that

$$(x_0)_\lambda\subset C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0},\lfloor \mathbf{n}_\lambda x_0\rfloor)$$

and conclude that $|C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0},\lfloor \mathbf{n}_\lambda x_0\rfloor)|\ge 2\mathbf{m}_\lambda$, whence

$$W^{\lambda,\pi}_{t_0}(x_0) \ge 1-\frac{\log(\mathbf{a}_\lambda)}{\log(1/\lambda)} \ge 1-\varepsilon.$$

Recalling that $Z_{t_0}(x_0)=(t_0-\tau_{t_0}(x_0))\wedge 1$, we have proved that

$$\mathbb{P}\left[|W^{\lambda,\pi}_{t_0}(x_0)-Z_{t_0}(x_0))|<\varepsilon\right] \ge \mathbb{P}\left[\Omega(\alpha_0,\gamma_0,\lambda,\pi)\cap\Omega_{A,T}^{x_0,t_0}(\alpha_0,\lambda,\pi)\right] \ge 1-\delta,$$

as desired.

Step 4. Here we prove (c). Recall that $Z^{\lambda,\pi}_{t_0}(x_0) = \left(-\frac{\log(1-K^{\lambda,\pi}_{t_0}(x_0))}{\log(1/\lambda)}\right) \wedge 1$ where $K^{\lambda,\pi}_{t_0}(x_0) = (2\mathbf{m}_\lambda+1)^{-1}\left|\left\{i \in [\![\lfloor \mathbf{n}_\lambda X_0\rfloor - \mathbf{m}_\lambda\,,\lfloor \mathbf{n}_\lambda X_0\rfloor + \mathbf{m}_\lambda]\!] : \eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) = 1\right\}\right|$. We work on $\Omega(\alpha_0,\gamma_0,\lambda,\pi) \cap \Omega^{x_0,t_0}_{A,T}(\alpha_0,\lambda,\pi)$ and set $t_c = t_0 - \tau_{t_0}(x_0)$.

Case 1. If $t_c \geq 1$, we have checked in Step 3 that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) = 1$ for all $i \in (x_0)_\lambda$, whence $K^{\lambda,\pi}_{t_0}(x_0) = 1$ and $Z^{\lambda,\pi}_{t_0}(x_0) = 1$.

Case 2. If now $0 < t_c < 1$, we deduce from Step 3 that

$$\underline{K}^{\lambda,\pi}_{t_0}(x_0) \leq K^{\lambda,\pi}_{t_0}(x_0) \leq \overline{K}^{\lambda,\pi}_{t_0}(x_0)$$

where

$$\begin{aligned}\underline{K}^{\lambda,\pi}_{t_0}(x_0) &= (2\mathbf{m}_\lambda+1)^{-1}\left|\left\{i \in [\![\lfloor \mathbf{n}_\lambda X_0\rfloor - \mathbf{m}_\lambda\,,\lfloor \mathbf{n}_\lambda x_0\rfloor + \mathbf{m}_\lambda]\!] : \underline{\eta}^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) = 1\right\}\right|,\\ \overline{K}^{\lambda,\pi}_{t_0}(x_0) &= (2\mathbf{m}_\lambda+1)^{-1}\left|\left\{i \in [\![\lfloor \mathbf{n}_\lambda X_0\rfloor - \mathbf{m}_\lambda\,,\lfloor \mathbf{n}_\lambda x_0\rfloor + \mathbf{m}_\lambda]\!] : \overline{\eta}^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) = 1\right\}\right|.\end{aligned}$$

Recalling Step 5 in Subsection II.8.5, we deduce that

$$\mathbb{P}\left[K^{\lambda,\pi}_{t_0}(x_0) \in (1-\lambda^{t_c-\varepsilon}\,,1-\lambda^{t_c+\varepsilon})\right] \geq 1 - c\delta,$$

for some constant $c > 0$, whence

$$\mathbb{P}\left[Z^{\lambda,\pi}_{t_0}(x_0) \in (t_c-\varepsilon\,,t_c+\varepsilon)\right] \geq 1 - c\delta.$$

This is nothing but the goal, since $Z_{t_0}(x_0) = t_0 - \tau_{t_0}(x_0) = t_c$ as soon as $Z_{t_0}(x_0) < 1$.

Step 5. It remains to prove (a). On $\Omega(\alpha_0,\gamma_0,\lambda,\pi) \cap \Omega^{x_0,t_0}_{A,T}(\alpha_0,\lambda,\pi)$, we check that

(i) If $Z_{t_0}(x_0) < 1$, then $D_{t_0}(x_0) = \{x_0\}$ and $C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}, \lfloor \mathbf{n}_\lambda x_0\rfloor) \subset (x_0)_\lambda$ (see Step 3 above), whence $D^{\lambda,\pi}_{t_0}(x_0) \subset [x_0 - \mathbf{m}_\lambda/\mathbf{n}_\lambda\,, x_0 + \mathbf{m}_\lambda/\mathbf{n}_\lambda]$. We deduce that

$$\boldsymbol{\delta}(D^{\lambda,\pi}_{t_0}(x_0), D_{t_0}(x_0)) \leq 2\mathbf{m}_\lambda/\mathbf{n}_\lambda.$$

(ii) If $Z_{t_0}(x_0) = 1$ and $D_{t_0}(x_0) = [a\,,b]$, for some $a, b \in \mathcal{B}_M \cup \{-A, A\}$, then

- for $c \in \mathcal{C}_M$ with $c \subset (a\,,b)$, $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) = 1$ for all $i \in c_\lambda$ (see Step 4 of the preceeding proof);
- for $x \in \mathcal{B}_M \cap (a\,,b)$, $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) = 1$ for all $i \in (x)_\lambda$ (see Step 5 of the preceeding proof);
- there are $i \in (a)_\lambda$ and $j \in (b)_\lambda$ such that $\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(i) = \eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0}(j) = 0$ (see Step 6 of the preceeding proof);

so that

$$[\![\lfloor \mathbf{n}_\lambda a\rfloor + \mathbf{m}_\lambda\,,\lfloor \mathbf{n}_\lambda b\rfloor - \mathbf{m}_\lambda]\!] \subset C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t_0},\lfloor \mathbf{n}_\lambda x_0\rfloor) \subset [\![\lfloor \mathbf{n}_\lambda a\rfloor - \mathbf{m}_\lambda\,,\lfloor \mathbf{n}_\lambda b\rfloor + \mathbf{m}_\lambda]\!]$$

and thus

$$[a+\mathbf{m}_\lambda/\mathbf{n}_\lambda\,, b-\mathbf{m}_\lambda/\mathbf{n}_\lambda] \subset D^{\lambda,\pi}_{t_0}(x_0) \subset [a-\mathbf{m}_\lambda/\mathbf{n}_\lambda\,, b+\mathbf{m}_\lambda/\mathbf{n}_\lambda],$$

whence $\boldsymbol{\delta}(D^{\lambda,\pi}_{t_0}(x_0), D_{t_0}(x_0)) \le 4\mathbf{m}_\lambda/\mathbf{n}_\lambda$.

Thus, on $\Omega(\alpha_0,\gamma_0,\lambda,\pi)\cap\Omega^{x_0,t_0}_{A,T}(\alpha_0,\lambda,\pi)$, we always have $\boldsymbol{\delta}(D^{\lambda,\pi}_{t_0}(x_0), D_{t_0}(x_0)) \le 4\mathbf{m}_\lambda/\mathbf{n}_\lambda$. We conclude that

$$\mathbb{P}\left[\boldsymbol{\delta}(D^{\lambda,\pi}_{t_0}(x_0), D_{t_0}(x_0)) \le \varepsilon\right] \ge \mathbb{P}\left[\Omega(\alpha_0,\gamma_0,\lambda,\pi)\cap\Omega^{x_0,t_0}_{A,T}(\alpha_0,\lambda,\pi)\right] \ge 1-\delta.$$

This concludes the proof. □

II.9.6. Cluster size distribution when $p=0$

The aim of this section is to prove Corollary II.2.6 when $p=0$. We first recall a result of [[BF13], Lemma 3.11.1].

Lemma II.9.5. *Let $(Z_t(x),H_t(x),F_t(x))_{t\ge0,x\in\mathbb{R}}$ be a LFFP(0) and consider $(D_t(x))_{t\ge0,x\in\mathbb{R}}$ the associated process. There are some constants $0<c_1<c_2$ and $0<\kappa_1<\kappa_2$ such that the following estimates hold.*

(i) For any $t\in(1,\infty)$, any $x\in\mathbb{R}$, any $z\in[0,1)$, $\mathbb{P}[Z_t(x)=z]=0$.

(ii) For any $t\in[0,\infty)$, any $B>0$, any $x\in\mathbb{R}$, $\mathbb{P}[|D_t(x)|=B]=0$.

(iii) For all $t\in[0,\infty)$, all $x\in\mathbb{R}$, all $B>0$, $\mathbb{P}[|D_t(x)|\ge B]\le c_2e^{-\kappa_1 B}$.

(iv) For all $t\in[\frac{3}{2},\infty)$, all $x\in\mathbb{R}$, all $B>0$, $\mathbb{P}[|D_t(x)|\ge B]\ge c_1e^{-\kappa_2 B}$.

(v) For all $t\in[5/2,\infty)$, all $0\le a<b<1$, all $x\in\mathbb{R}$,

$$c_1(b-a)\le\mathbb{P}[Z_t(x)\in[a\,,b]]\le c_2(b-a).$$

We now handle the

Proof of Corollary II.2.6 when $p=0$. For each $\lambda\in(0\,,1)$ and each $\pi\ge1$, consider a $(\lambda,\pi)-$FFP $(\eta^{\lambda,\pi}_t(i))_{t\ge0,i\in\mathbb{Z}}$. Let also $(Z_t(x),H_t(x),F_t(x))_{t\ge0,x\in\mathbb{R}}$ be a LFFP(0) and consider the corresponding process $(D_t(x))_{t\ge0,x\in\mathbb{R}}$.

Point (b). Using Lemma II.9.5-(iii)-(iv) and recalling that $|C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t},0)|/\mathbf{n}_\lambda = |D^{\lambda,\pi}_t(0)|$, it suffices to check that for all $t\ge3/2$ and all $B>0$, when $\lambda\to0$ and $\pi\to0$ in the regime $\mathcal{R}(0)$,

$$\lim_{\lambda,\pi}\mathbb{P}\left[|D^{\lambda,\pi}_t(0)|\ge B\right]=\mathbb{P}[|D_t(0)|\ge B].$$

This follows from Theorem II.2.4-2, which implies that $|D^{\lambda,\pi}_t(0)|$ goes in law to $|D_t(0)|$ and from Lemma II.9.5-(ii).

Point (a). Due to Lemma II.9.5-(v) we only need that for all $0 < a < b < 1$, all $t \geq 5/2$, when $\lambda \to 0$ and $\pi \to 0$ in the regime $\mathcal{R}(0)$,

$$\lim_{\lambda,\pi} \mathbb{P}\left[|C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}, 0)| \in [\lambda^{-a}\,, \lambda^{-b}]\right] = \mathbb{P}\left[Z_t(0) \in [a\,, b]\right].$$

But using Theorem II.2.4-3 and Lemma II.9.5-(i), we know that

$$\lim_{\lambda,\pi} \mathbb{P}\left[\frac{\log(|C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}, 0)|)}{\log(1/\lambda)} \mathbf{1}_{\{|C(\eta^{\lambda,\pi}_{\mathbf{a}_\lambda t}, 0)| \geq 1\}} \in [a\,, b]\right] = \mathbb{P}\left[Z_t(0) \in [a\,, b]\right]$$

as $\lambda \to 0$ and $\pi \to 0$ in the regime $\mathcal{R}(0)$. One immediately concludes. $\square$

III. Asymptotic of the one dimensional forest-fire processes in random media

Sommaire

Résumé

On considère le modèle suivant des feux de forêts sur $\mathbb{Z}$, où chaque site a deux états possibles : *vide* ou *occupé*. Donnons nous un paramètre $\lambda > 0$, une loi ν sur $(0, \infty)$ et une suite $(\kappa_i)_{i\in\mathbb{Z}}$ de variables aléatoires indépendantes identiquement distribuées selon ν. Un site vide i devient occupé avec taux κ_i. Sur chaque site, des allumettes tombent avec taux λ et détruisent immédiatement la composante de sites occupés correspondante. On étudie l'asymptotique des feux rares. Sous une hypothèse raisonnable sur ν, on espère que le processus converge, avec une renormalisation correcte, vers un modèle limite. On s'attend à distinguer trois processus limites différents.

Abstract

Consider the following forest fire model where the possible locations of trees are the sites of $\mathbb{Z}$. Each site has two possible states: 'vacant' or 'occupied'. Consider a law ν on $(0, \infty)$ and an i.i.d. sequence of random variables $(\kappa_i)_{i\in\mathbb{Z}}$ with law ν. Each vacant site i becomes occupied at rate κ_i. At each site, ignition (by lightning) occurs at rate λ. When a site is ignited, a fire starts and destroys immediately the corresponding connected component of occupied sites. We study the asymptotic behavior of this process as $\lambda \to 0$. Under some quite reasonable assumptions on the law ν, we hope that the process converges, with a correct normalization, to a limit forest fire model. We expect that there are three possible classes of scaling limits.

III.1. Definitions, notation and assumptions

III.1.1. The discrete model

For $a, b \in \mathbb{Z}$, we set $[\![a\,,b]\!] = \{a, \ldots, b\} \subset \mathbb{Z}$. For $\eta \in \{0,1\}^{\mathbb{Z}}$ and $i \in \mathbb{Z}$, we define the occupied connected component around i as

$$C(\eta, i) = \begin{cases} \emptyset & \text{if } \eta(i) = 0, \\ [\![l(\eta,i)\,, r(\eta,i)]\!] & \text{if } \eta(i) = 1, \end{cases}$$

where $l(\eta, i) = \sup\{k < i : \eta(k) = 0\} + 1$ and $r(\eta, i) = \inf\{k > i : \eta(k) = 0\} - 1$.

Definition III.1.1. *Let $\lambda \in (0\,,1]$, ν a probability distribution on $(0\,,\infty)$ and $(\kappa_i)_{i\in\mathbb{Z}}$ an i.i.d. sequence of random variables with law ν. For each $i \in \mathbb{Z}$, consider two Poisson processes $N^S(i) = (N_t^S(i))_{t\geq 0}$ and $N^M(i) = (N_t^M(i))_{t\geq 0}$ with respective parameters κ_i and λ, all these processes being independent. Consider a $\{0,1\}^{\mathbb{Z}}-$valued process such that a.s., for all $i \in \mathbb{Z}$, the process $(\eta_t^\lambda(i))_{t\geq 0}$ is càdlàg. We say that $(\eta_t^\lambda(i))_{t\geq 0, i\in\mathbb{Z}}$ is a $(\lambda,\nu)-$Forest Fire Process in Random Media ($(\lambda,\nu)-$FFPRM in short) if a.s., for all $t \geq 0$ and all $i \in \mathbb{Z}$,*

$$\eta_t^\lambda(i) = \int_0^t \mathbf{1}_{\{\eta_{s-}^\lambda(i)=0\}}\, \mathrm{d}N_s^S(i) - \sum_{k\in\mathbb{Z}} \int_0^t \mathbf{1}_{\{k\in C(\eta_{s-}^\lambda, i)\}}\, \mathrm{d}N_s^M(k).$$

Formally, we say that $\eta_t^\lambda(i) = 0$ if there is no tree at site i at time t and $\eta_t^\lambda(i) = 1$ if the site i is occupied. Thus, the forest fire process starts from an empty initial configuration, on each site i, seeds fall according to some Poisson process of parameter κ_i and matches fall according to some Poisson process of parameter λ. When a seed falls on an empty site, a tree appears immediately. When a match falls on an occupied site, it burns instantaneously the corresponding connected component of occupied sites. Seeds falling on occupied sites and matches falling on vacant sites have no effect. This process can be shown to exist and to be unique (for almost every realization of N^S, N^M) by using a graphical construction. Indeed, to build the process until a given time $T > 0$, it suffices to work between sites i which are vacant until time T [because $N_T^S(i) = 0$]. Interaction cannot cross such sites.

III.1.2. Assumption

Our assumptions will concern the Laplace transform of ν.

Definition III.1.2. *The Laplace transform of the law ν on $(0\,,\infty)$ is defined as*

$$G(t) = \int_0^\infty e^{-xt}\nu(\mathrm{d}x).$$

Observe that G is convex, non increasing and $G(t) \xrightarrow[t\to\infty]{} 0$.

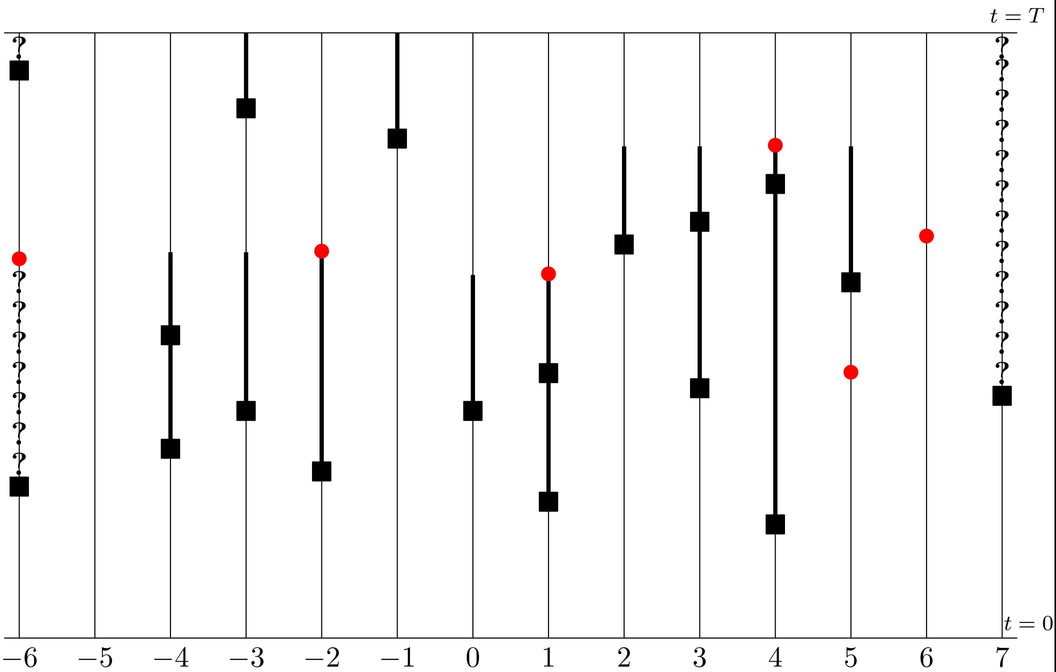

Figure III.1.: Graphical construction of the $(\lambda, \nu)-$FFPRM.

Matches are represented as bullets and seeds as squares. On the sites -5 and 6, no seed fall during $[0\,,T]$, so that these sites remain vacant until T. One can thus clearly deduce the values of the process in $[\![-5\,,6]\!]$ during $[0\,,T]$ using only the bullets and squares inside $[\![-5\,,6]\!]$.

In the rest of the paper, we will assume that the Laplace transform of the law ν satisfies

$$\forall t > 0,\ \lim_{x\to\infty} \frac{G(x)}{G(xt)} \in [0\,,\infty) \cup \{\infty\} \text{ exists.} \tag{III.1.1}$$

It is well known ([Kor04], Theorem 2.3 p. 181) that, if (III.1.1) holds true, then there is $\beta \in [0\,,\infty) \cup \{\infty\}$ such that

RV(β)**:** The Laplace transform of the law ν satisfies

$$\forall t > 0,\ \lim_{x\to\infty} \frac{G(x)}{G(xt)} = t^\beta, \tag{III.1.2}$$

with the convention

$$t^\infty = \begin{cases} 0 & \text{if } t \in (0\,,1), \\ 1 & \text{if } t = 1, \\ \infty & \text{if } t \in (1\,,\infty). \end{cases}$$

When $\beta > 0$, we say that $1/G$ is a regularly varying function with index β. When $\beta = 0$, $1/G$ is said to be a slowly varying function whereas $1/G$ is said to be a rapidly varying function when $\beta = \infty$.

Observe that this hypothesis is not so restrictive, since it is satisfied by all reasonable laws (G is decreasing, convex and analytic on $(0\,,\infty)$).

Roughly, under $\mathbf{RV}(\infty)$, all the κ_i's are not *too small i.e.* they are not too close to 0. This happens for example when there is $a_0 > 0$ such that $\nu([a_0\,,\infty)) = 1$. However, there are some laws ν such that $\nu((\varepsilon\,,\infty)) < 1$ for all $\varepsilon > 0$ with Laplace transform satisfying $\mathbf{RV}(\infty)$, for example the law of an $\alpha-$stable subordinator, with $\alpha \in (0\,,1)$, which satisfies $G(t) = e^{-t^\alpha}$ for all $t \geq 0$. In this case, there are *very few* κ_i's which are close to 0.

The archetype of law which satisfy $\mathbf{RV}(\beta)$, for $\beta \in (0\,,\infty)$, is the Gamma distribution which has for density $f_\beta(x) = (x^{\beta-1}e^{-x}/\Gamma(\beta))\mathbf{1}_{\{x>0\}}$.

We finally introduce the following notation.

Notation III.1.3. *We set*

$$\varphi(t) = \int_0^t \frac{1}{G(s)}\,\mathrm{d}s \tag{III.1.3}$$

and we define ψ as the inverse function of φ. Clearly, φ is non-decreasing, $\varphi(0) = 0$ and $\lim_{t\to\infty}\varphi(t) = \infty$ *and ψ has the same properties.*

III.1.3. Notation

In the whole paper, we denote, for $I \subset \mathbb{Z}$, by $|I| = \#I$ the number of elements in I. For $I = [\![a\,,b]\!] = \{a,\dots,b\} \subset \mathbb{Z}$ and $\alpha > 0$, we will set $\alpha I \coloneqq [\alpha a\,,\alpha b] \subset \mathbb{R}$. For $\alpha > 0$, we of course take the convention that $\alpha\emptyset = \emptyset$.

For $J = [a\,,b]$ an interval of $\mathbb{R}$, $|J| = b - a$ stands for the length of J and for $\alpha > 0$, we set $\alpha J = [\alpha a\,,\alpha b]$.

For $x \in \mathbb{R}$, $\lfloor x \rfloor$ stands for the integer part of x.

We denote by $\mathcal{I} = \{[a\,,b] : a \leq b\}$ the set of all closed finite intervals of $\mathbb{R}$. For two intervals $[a\,,b]$ and $[c\,,d]$, we set

$$\begin{aligned} \boldsymbol{\delta}([a\,,b],[c\,,d]) &= |a-c| + |b-d|, \\ \boldsymbol{\delta}([a\,,b],\emptyset) &= |a-b|. \end{aligned} \tag{III.1.4}$$

For two functions $I, J\colon [0\,,T] \to \mathcal{I}\cup\{\emptyset\}$, we set

$$\boldsymbol{\delta}_T(I,J) = \int_0^T \boldsymbol{\delta}(I_t,J_t)\,\mathrm{d}t. \tag{III.1.5}$$

For $(x,I),(y,J)$ in $\mathbb{D}([0\,,T],\mathbb{R}_+\times(\mathcal{I}\cup\{\emptyset\}))$, the set of càdlàg functions from $[0\,,T]$ into $\mathbb{R}_+\times\mathcal{I}\cup\{\emptyset\}$, we define

$$\mathbf{d}_T((x,I),(y,J)) = \sup_{t\in[0,T]} |x(t)-y(t)| + \boldsymbol{\delta}_T(I,J). \tag{III.1.6}$$

III.2. Heuristic scales and relevant quantities

III.2.1. Time and space scales

We look for some time scale for which tree clusters see about one fire per unit of time. But for λ very small, clusters will be very large before a match falls inside. We thus also have to rescale space.

Time scale

For $\lambda > 0$ very small and for t not too large, one might neglect fires, so that roughly, each site i is vacant with probability $\mathbb{E}\left[e^{-\kappa_i t}\right] = G(t)$ (because the time we have to wait for the first seed follows, on each site, the law $\mathcal{E}(\kappa_i)$). Consequently, for t not too large,

$$\mathbb{E}\left[|C(\eta_t^\lambda, 0)|\right] \simeq \frac{2}{G(t)}. \tag{III.2.1}$$

On the other hand, the rate that at which matches fall in the cluster $C(\eta_t^\lambda, 0)$ is $\lambda|C(\eta_t^\lambda, 0)|$.

We decide to accelerate time by a factor $\mathbf{a}_\lambda$ which solves

$$\lambda \int_0^{\mathbf{a}_\lambda} \frac{1}{G(s)}\,\mathrm{d}s = \lambda\varphi(\mathbf{a}_\lambda) = 1. \tag{III.2.2}$$

By the way, the probability that a match falls in $C(\eta^\lambda, 0)$ during $[0, \mathbf{a}_\lambda]$ should tend to some nontrivial value.

Observe that

$$\mathbf{a}_\lambda \xrightarrow[\lambda\to 0]{} \infty, \tag{III.2.3}$$

$$\lambda\mathbf{a}_\lambda \xrightarrow[\lambda\to 0]{} 0. \tag{III.2.4}$$

Indeed, recall Notation III.1.3. We have $\mathbf{a}_\lambda = \psi(1/\lambda)$. Clearly, $\lambda \mapsto \mathbf{a}_\lambda$ is non-increasing and tends to ∞ as $\lambda \to 0$. Next, since $G(t)$ decreases to 0 as $t \to \infty$, we easily deduce that $\varphi(t)/t$ increases to ∞ as $t \to \infty$. Consequently, $\varphi(\mathbf{a}_\lambda)/\mathbf{a}_\lambda$ tends to ∞ as $\lambda \to 0$, which implies, since $\varphi(\mathbf{a}_\lambda) = 1/\lambda$, that $\lambda\mathbf{a}_\lambda \to 0$ as $\lambda \to 0$.

Space scale

We now rescale space in such a way that during a time interval of order $\mathbf{a}_\lambda$, something like one match falls per unit of (space) length. Since fires occur at rate λ, our space scale has to be of order

$$\mathbf{n}_\lambda = \left\lfloor \frac{1}{\lambda\mathbf{a}_\lambda} \right\rfloor. \tag{III.2.5}$$

This means that we will identify $[\![0, \mathbf{n}_\lambda]\!] \subset \mathbb{Z}$ with $[0, 1] \subset \mathbb{R}$.

III.2.2. Rescaled cluster

We thus set, for $\lambda \in (0,1)$, $t \geq 0$ and $x \in \mathbb{R}$,

$$D_t^\lambda(x) := \frac{1}{\mathbf{n}_\lambda} C(\eta^\lambda_{\mathbf{a}_\lambda t}, \lfloor \mathbf{n}_\lambda x \rfloor) \subset \mathbb{R}. \tag{III.2.6}$$

Using (III.2.1) and Lemma A.1, we see that

$$\left| D_t^\lambda(x) \right| \simeq \frac{2}{\mathbf{n}_\lambda G(\mathbf{a}_\lambda t)} \xrightarrow[\lambda \to 0]{} \begin{cases} 2(\beta+1)t^\beta & \text{if } \beta \in [0,\infty) \\ t^\infty & \text{if } \beta = \infty. \end{cases} \tag{III.2.7}$$

Case $\beta \in [0,\infty)$.

In this case, everything is fine: for all times of order $\mathbf{a}_\lambda t$, the good space scale is indeed $\mathbf{n}_\lambda$. Thus we will describe the $(\lambda,\nu)-$FFPRM through $(D_t^\lambda(x))_{t \geq 0, x \in \mathbb{R}}$.

Case $\beta = \infty$.

This estimate (III.2.7) (neglecting fires) suggests that for all $x \in \mathbb{R}$, for $t < 1$, $|D_t^\lambda(x)| \to 0$ and for $t > 1$, $|D_t^\lambda(x)| \to \infty$. For $t > 1$, fires might be in effect and we hope that this will make finite the possible limit of $|D_t^\lambda(x)|$. But fires can only reduce the size of clusters, so that for $t < 1$, the limit of $|D_t^\lambda(x)|$ will really be 0.

Since we would like to have an idea of the sizes of microscopic clusters, we have to keep some information about the degree of smallness of microscopic clusters.

We consider a function $\mathbf{m}_\lambda \colon (0,1] \to \mathbb{N}$ satisfying

$$\begin{cases} \lim_{\lambda\to 0} \mathbf{m}_\lambda = \infty, \ \lim_{\lambda \to 0}(\mathbf{m}_\lambda/\mathbf{n}_\lambda) = 0, \\ \lambda \mapsto \mathbf{m}_\lambda \text{ is non-increasing}, \\ \forall z \in [0,1), \lim_{\lambda \to 0} \mathbf{m}_\lambda G(\mathbf{a}_\lambda z) = \infty, \\ (2\mathbf{m}_\lambda + 1) G(\mathbf{a}_\lambda) < 1. \end{cases} \tag{III.2.8}$$

The existence of such a function will be proved in Lemma A.3.

We introduce, for $\lambda > 0$, $x \in \mathbb{R}$ and $t > 0$,

$$K_t^\lambda(x) := \frac{\left| \left\{ i \in [\![\lfloor \mathbf{n}_\lambda x \rfloor - \mathbf{m}_\lambda , \lfloor \mathbf{n}_\lambda x \rfloor + \mathbf{m}_\lambda]\!] : \eta^\lambda_{\mathbf{a}_\lambda t}(i) = 1 \right\} \right|}{2\mathbf{m}_\lambda + 1} \in [0,1], \tag{III.2.9}$$

$$Z_t^\lambda(x) := \frac{G^{-1}(1 - K_t^\lambda(x))}{\mathbf{a}_\lambda} \wedge 1 \in [0,1]. \tag{III.2.10}$$

Observe that $K_t^\lambda(x)$ stands for the local density of occupied sites around $\lfloor \mathbf{n}_\lambda x \rfloor$ at time $\mathbf{a}_\lambda t$. This density is local because $\mathbf{m}_\lambda \ll \mathbf{n}_\lambda$. We hope that for $t < 1$, neglecting fires,

$$K_t^\lambda(x) \simeq 1 - G(\mathbf{a}_\lambda t),$$

because each site is occupied at time $\mathbf{a}_\lambda t$ with probability $1 - G(\mathbf{a}_\lambda t)$, whence $Z_t^\lambda(x) \simeq t$.

It also holds that $Z^\lambda_t(x) = 1$ if and only if $K^\lambda_t(x) = 1$, *i.e.* if and only if all the sites in $[\![\lfloor \mathbf{n}_\lambda x\rfloor - \mathbf{m}_\lambda\,, \lfloor \mathbf{n}_\lambda x\rfloor + \mathbf{m}_\lambda]\!]$ are occupied. Indeed, $Z^\lambda_t(x) = 1$ implies that $G^{-1}(1 - K^\lambda_t(x)) \geq \mathbf{a}_\lambda$, so that

$$K^\lambda_t(x) \geq 1 - G(\mathbf{a}_\lambda) > 1 - \frac{1}{2\mathbf{m}_\lambda + 1},$$

whence $K^\lambda_t(x) = 1$. This last assertion comes from the facts that $K^\lambda_t(x)$ takes its values in $\{k/(2\mathbf{m}_\lambda + 1) : k = 0, \dots, 2\mathbf{m}_\lambda + 1\}$.

Since we will allow $\mathbf{m}_\lambda$ to be arbitrarily close to $\mathbf{n}_\lambda$, $Z^\lambda_t(x) = 1$ will imply, roughly, that the cluster containing $\lfloor \mathbf{n}_\lambda x\rfloor$ is macroscopic, *i.e.* has a length of order $\mathbf{n}_\lambda$.

We will study the $(\lambda, \nu)-$FFPRM through $(D^\lambda_t(x), Z^\lambda_t(x))_{t\geq 0, x\in\mathbb{R}}$. The main idea is that for $\lambda > 0$ very small:

- If $Z^\lambda_t(x) = z \in (0\,,1)$, then $|D^\lambda_t(x)| \simeq 0$ and the (rescaled) cluster containing x is microscopic (in the sense that the non-rescaled cluster is small when compared to $\mathbf{n}_\lambda$), but we control the local density of occupied sites around x, which resembles $1 - G(\mathbf{a}_\lambda z)$. Observe that this density tends to 1 as $\lambda \to 0$ for all $z \in (0\,,1)$.

- If $Z^\lambda_t(x) = 1$ and $D^\lambda_t(x) = [a\,,b]$ then the (rescaled) cluster containing x is macroscopic and has a length equal to $b - a$, or

  $$|C(\eta^\lambda_{\mathbf{a}_\lambda t}, \lfloor \mathbf{n}_\lambda x\rfloor)| \simeq (b - a)\mathbf{n}_\lambda$$

  in the original scales.

Comparing the heuristic description above with the heuristic description given in [BF10], the limit process as $\lambda \to 0$ for the $(\lambda, \nu)-$FFPRM should be the same as in [BF10].

Summary

- We accelerate time by the factor $\mathbf{a}_\lambda$, defined by $\lambda\varphi(\mathbf{a}_\lambda) = 1$.

- Our space scale is $\mathbf{n}_\lambda = \lfloor 1/(\lambda \mathbf{a}_\lambda)\rfloor$.

- If $\beta \in [0\,,\infty)$, we will only study the rescaled clusters $(D^\lambda_t(x))_{t\geq 0, x\in\mathbb{R}}$.

- If $\beta = \infty$, we will study the rescaled clusters $(D^\lambda_t(x))_{t\geq 0, x\in\mathbb{R}}$, as well as the local densities of occupied sites $(Z^\lambda_t(x))_{t\geq 0, x\in\mathbb{R}}$.

III.3. The case $\beta = \infty$

III.3.1. Definition of the limit process

We describe the limit process in the case where $\beta = \infty$. As mentioned above, it is exactly the same process as in the Poisson case studied in [BF10] and is well interstood. We consider a Poisson measure $\pi_M(\mathrm{d}x, \mathrm{d}t)$ on $\mathbb{R} \times [0\,,\infty)$, with intensity measure $\mathrm{d}x\,\mathrm{d}t$, whose marks correspond to matches.

Definition III.3.1. *A process $(Z_t(x), D_t(x), H_t(x))_{t\geq 0, x\in\mathbb{R}}$ with values in $\mathbb{R}_+ \times \mathcal{I} \times \mathbb{R}_+$ such that a.s., for all $x \in \mathbb{R}$, $(Z_t(x), H_t(x))_{t\geq 0}$ is càdlàg, is said to be a LFF$(\infty)-$process if a.s., for all $t \geq 0$, all $x \in \mathbb{R}$,*

$$Z_t(x) = \int_0^t \mathbf{1}_{\{Z_s(x)<1\}}\,\mathrm{d}s - \int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{Z_{s-}(x)=1, y\in D_{s-}(x)\}} \pi_M(\mathrm{d}s, \mathrm{d}y),$$
$$H_t(x) = \int_0^t Z_{s-}(x) \mathbf{1}_{\{Z_s(x)<1\}} \pi_M(\mathrm{d}s \times \{x\}) - \int_0^t \mathbf{1}_{\{H_s(x)>0\}}\,\mathrm{d}s,$$

where $D_t(x) = [L_t(x)\,, R_t(x)]$, with

$$L_t(x) = \sup\left\{y \leq x : Z_t(y) < 1 \text{ or } H_t(y) > 0\right\},$$
$$R_t(x) = \inf\left\{y \geq x : Z_t(y) < 1 \text{ or } H_t(y) > 0\right\}$$

and where $D_{t-}(x)$ is defined in the same way.

We refer to [BF10] for the formal dynamic of this process.

III.3.2. Well-posedness

The existence and uniqueness of the LFF$(\infty)-$process has already been proved in [BF10].

Theorem III.3.2. *For any Poisson measure $\pi_M(\mathrm{d}x, \mathrm{d}t)$ on $\mathbb{R} \times [0\,, \infty)$ with intensity measure $\mathrm{d}x\,\mathrm{d}t$, there a.s. exists a unique LFF$(\infty)-$process. Furthermore, it can be constructed graphically and its restriction to any finite box $[-n\,, n] \times [0\,, T]$ can be perfectly simulated.*

III.3.3. The convergence result.

We expect the following Theorem. We will give a heuristic proof in the next Section.

Theorem III.3.3. *Let ν be a probability distribution on $(0\,, \infty)$ and $(\kappa_i)_{i\in\mathbb{Z}}$ an i.i.d. sequence of random variables with law ν. For each $\lambda \in (0, 1)$, consider the process $(Z_t^\lambda(x), D_t^\lambda(x))_{t\geq 0, x\in\mathbb{R}}$ associated with the $(\lambda, \nu)-$FFPRM. Consider also the LFF$(\infty)-$process $(Z_t(x), D_t(x), H_t(x))_{t\geq 0, x\in\mathbb{R}}$. Assume* $\mathbf{RV}(\infty)$.

1. *For any $T > 0$, any finite subset $\{x_1, \dots, x_p\} \subset \mathbb{R}$,*

$$(Z_t^\lambda(x_i), D_t^\lambda(x_i))_{t\in[0,T], i=1,\dots,p}$$

goes in law to $(Z_t(x_i), D_t(x_i))_{t\in[0,T], i=1,\dots,p}$ in $\mathbb{D}([0\,, T], \mathbb{R} \times \mathcal{I} \cup \emptyset)^p$, as λ tends to 0. Here $\mathbb{D}([0\,, T], \mathbb{R} \times \mathcal{I} \cup \emptyset)$ is endowed with the distance $\mathbf{d}_T$.

2. *For any finite subset $\{(x_1, t_1), \dots, (x_p, t_p)\} \subset \mathbb{R} \times [0, \infty)$, with $t_k \neq 1$ for $k = 1, \dots, p$, $(Z_{t_i}^\lambda(x_i), D_{t_i}^\lambda(x_i))_{i=1,\dots,p}$ goes in law to $(Z_{t_i}(x_i), D_{t_i}(x_i))_{t\in[0,T], i=1,\dots,p}$ in $(\mathbb{R} \times \mathcal{I} \cup \emptyset)^p$, as λ tends to 0. Here $(\mathbb{R} \times \mathcal{I} \cup \emptyset)$ is endowed with the distance $\boldsymbol{\delta}$.*

3. Using the convention $G^{-1}(1/0) = 0$, for all $t > 0$,

$$\left(\frac{G^{-1}(1/|C(\eta^{\lambda}_{\mathbf{a}_\lambda t}, 0)|)}{\mathbf{a}_\lambda}\right) \wedge 1$$

goes in law to $Z_t(0)$ as $\lambda \to 0$.

Observe that the random media has disappeared in the limit process.

Example 1. Consider the case where there is $a_0 > 0$ such that $\inf(\operatorname{supp} \nu) = a_0$. Here, G clearly satisfies $\mathbf{RV}(\infty)$ and we have (see Appendix A, Example 1)

$$\mathbf{a}_\lambda \sim \frac{1}{a_0} \log(1/\lambda).$$

Example 2. Here we examine the example where $G(t) = e^{-t^\alpha}$ for all $t \geq 0$ and for some $\alpha \in (0, 1)$: G is the Laplace transform of the law of an $\alpha-$stable subordinator, which is supported by $(0, \infty)$. Observe that G satisfies $\mathbf{RV}(\infty)$. In this case, we have (see Appendix A, Example 2)

$$\mathbf{a}_\lambda \sim \log(1/\lambda)^{1/\alpha}.$$

Remark III.3.4. *Let us consider*

$$\nu = \theta\delta_{a_0} + (1-\theta)\delta_{b_0},$$

with $0 < a_0 < b_0$ and $\theta \in (0, 1)$. In this case, we have $\mathbf{a}_\lambda = \log(1/\lambda)/a_0$, see Example 1 above. It might look surprising at the first glance that neither the time and space scales $\mathbf{a}_\lambda$ and $\mathbf{n}_\lambda$ nor the limit process depend on the parameters θ and b_0. Only the definition of the process $(Z^\lambda_t(x))_{t\geq 0, x\in\mathbb{R}}$ depends on the parameters.

In fact, there are two kinds of sites. On the one hand, seeds fall often on sites i with $\kappa_i = b_0$. For example, at time $a_0/b_0 < 1$ (or at time $\log(1/\lambda)/b_0$ in the original scale), neglecting fires, all the sites i with $\kappa_i = b_0$ are occupied while sites i with $\kappa_i = a_0$ are all occupied only at time 1 (or $\log(1/\lambda)/a_0$ in the original scale). On the other hand, since sites i with $\kappa_i = a_0$ are uniformly distributed at random on $\mathbb{Z}$ (because the sequence $(\kappa_i)_{i\in\mathbb{Z}}$ is i.i.d.), in each zone of the form $[\![L, R]\!]$, with $L, R \in \mathbb{Z}$, $L < R$, there are roughly $\theta(R-L)$ slow *sites. But, using (III.2.8), we see that, for all $t, s > 0$,*

$$\mathbb{P}\left[\forall i \in [\![-\lfloor\theta\mathbf{m}_\lambda\rfloor, \lfloor\theta\mathbf{m}_\lambda\rfloor]\!], N^S_{\mathbf{a}_\lambda(t+s)}(i) - N^S_{\mathbf{a}_\lambda t}(i) > 0\right] \simeq (1 - G(\mathbf{a}_\lambda s))^{2\theta\mathbf{m}_\lambda}$$
$$\simeq \exp\left(-2\theta\mathbf{m}_\lambda G(\mathbf{a}_\lambda s)\right) \xrightarrow[\lambda\to 0]{} \begin{cases} 0 & \text{if } s < 1, \\ 1 & \text{if } s > 1. \end{cases}$$

Thus, microscopic zones, i.e. zone of the form $[\![-\mathbf{m}_\lambda, \mathbf{m}_\lambda]\!]$, become to macroscopic at time 1 (or at time $\log(1/\lambda)/a_0$ in the original scale) and one can neglect fast *sites i.e. one can consider that all the fast sites are always occupied.*

The same arguments show that any macroscopic zone destroyed by a fire will need a time exactly one to be completely occupied again.

Concerning microscopic fires, they will burn a larger zone if θ is small and/or b_0 is large. But the delay needed for this zone to be occupied again will always be (roughly) the same (because if θ is small and/or b_0 is large, more sites are fast*).*

III.3.4. Heuristic arguments

Let us explain here roughly the reasons why Theorem III.3.3 holds true. We consider, for $(\kappa_i)_{i\in\mathbb{Z}}$ a sequence of i.i.d. random variables with law ν and for $\lambda > 0$ very small, a $(\lambda,\nu)-$FFPRM $(\eta_t^\lambda(i))_{t\geq 0, i\in\mathbb{Z}}$ and the associated processes $(Z_t^\lambda(x), D_t^\lambda(x))_{t\geq 0, x\in\mathbb{R}}$.

1. Initial condition. For all $x \in \mathbb{R}$, $(Z_0^\lambda(x), D_0^\lambda(x)) = (0, \emptyset) \simeq (0, \{x\})$.

2. Occupation of vacant zones. Assume that a zone $[a\,, b]$ becomes completely vacant at some time t (because it has been destroyed by a fire).

(i) For $s \in [0\,, 1)$ and if no fire starts on $[a\,, b]$ during $[t\,, t + s]$, we have

$$D_{t+s}^\lambda(x) \simeq [x \pm 1/(\mathbf{n}_\lambda G(\mathbf{a}_\lambda s))] \simeq \{x\},$$

see Lemma A.1, and $Z_{t+s}^\lambda(x) \simeq s$ for all $x \in [a\,, b]$.

Indeed, $D_{t+s}^\lambda(x) \simeq [x - X/\mathbf{n}_\lambda\,, x + Y/\mathbf{n}_\lambda]$, where X and Y are approximately geometric random variables with parameter $G(\mathbf{a}_\lambda s)$. (Recall that for any $t, s \geq 0$ and for any site i, the probability that a seed fall in i during $[\mathbf{a}_\lambda t\,, \mathbf{a}_\lambda(t + s)]$ is $\mathbb{E}\left[1 - e^{-\mathbf{a}_\lambda \kappa_i s}\right] = 1 - G(\mathbf{a}_\lambda s)$). Thus

$$K_{t+s}^\lambda(x) \simeq 1 - G(\mathbf{a}_\lambda s),$$

whence $Z_{t+s}^\lambda(x) \simeq s$. For the same reasons, it holds that

$$D_{t+s}^\lambda(x) \simeq [x \pm 1/(\mathbf{n}_\lambda G(\mathbf{a}_\lambda s))] \simeq \{x\}$$

since $\mathbf{n}_\lambda G(\mathbf{a}_\lambda s) \to \infty$ because $s < 1$, recall Lemma (A.1).

(ii) If no fire starts on $[a\,, b]$ during $[t\,, t+1]$, then $Z_{t+1}^\lambda(x) \simeq 1$ and all the sites in $[a\,, b]$ are occupied (with very high probability) just after time $t + 1$.

Indeed, we have $(b-a)\mathbf{n}_\lambda$ sites and each of them is occupied at time $t+1+\varepsilon$ with approximate probability $1 - G(\mathbf{a}_\lambda(1 + \varepsilon))$, so that all of them are occupied with approximate probability

$$(1 - G(\mathbf{a}_\lambda(1 + \varepsilon))^{(b-a)\mathbf{n}_\lambda} \simeq \exp(-(b - a)\mathbf{n}_\lambda G(\mathbf{a}_\lambda(1 + \varepsilon)),$$

which tends to 1 as $\lambda \to 0$ for any $\varepsilon > 0$, thanks to Lemma A.1.

3. Microscopic fires. Assume that a fire starts at some place x at some time t, with $Z^\lambda_{t-}(x) = z \in (0\,,1)$. Then the possible clusters on the left and right of x cannot be connected during (approximately) $[t\,,t+z]$, but can be connected after (approximately) $t+z$.

Indeed, the match falls in a zone with approximate density $1 - G(\mathbf{a}_\lambda z)$, so that it should destroy a zone A of approximate length $1/G(\mathbf{a}_\lambda z) \ll \mathbf{n}_\lambda$. The probability that a fire starts again in A after t is very small. Thus the probability that A is completely occupied at time $t+s$ is approximately equal to

$$(1 - G(\mathbf{a}_\lambda s))^{1/G(\mathbf{a}_\lambda z)} \simeq \exp(-G(\mathbf{a}_\lambda s)/G(\mathbf{a}_\lambda z)).$$

When $\lambda \to 0$, this quantity tends to 0 if $s < z$ and to 1 if $s > z$.

4. Macroscopic fires. Assume now that a fire starts at some place x, at some time t and that $Z^\lambda_{t-}(x) \simeq 1$, so that $D^\lambda_{t-}(x)$ is macroscopic (that is its length is of order 1 in our scales, or of order $\mathbf{n}_\lambda$ in the original process). This will thus make vacant the zone $D^\lambda_{t-}(x)$. Such a (macroscopic) zone needs a time of order 1 to be completely occupied, see Point 2.

5. Clusters. For $t \geq 0$, $x \in \mathbb{R}$, the cluster $D^\lambda_t(x)$ resembles

$$[x \pm 1/(\mathbf{n}_\lambda G(\mathbf{a}_\lambda s))] \simeq \{x\}$$

if $Z^\lambda_t(x) = z \in (0\,,1)$, thanks to Lemma A.1. We then say that x is *microscopic*. *Macroscopic* clusters are delimited either by microscopic zones, or by sites where there has been recently a microscopic fire.

6. Random media. There is a slight abuse in the above arguments, since we more and less do as if on each site, seeds fall according to a renewal process of which delay's law has G for Laplace transform. However, this is not a true problem.

For example, let us check that, for all $a < b$ and all $0 < t_1 < t_2 < t_3 < t_4$, the probability that at least one seed falls on each site of $[\![\lfloor a\mathbf{n}_\lambda \rfloor\,, \lfloor b\mathbf{n}_\lambda \rfloor]\!]$ during the time intervals $[\mathbf{a}_\lambda t_1\,, \mathbf{a}_\lambda t_2]$ and $[\mathbf{a}_\lambda t_3\,, \mathbf{a}_\lambda t_4]$ tends to

$$\begin{cases} 1 & \text{if } t_2 - t_1 > 1 \text{ and } t_3 - t_4 > 1, \\ 0 & \text{if } t_2 - t_1 < 1 \text{ or } t_3 - t_4 < 1 \end{cases}$$

whereas, for all $z \in (0\,,1)$, the probability that at least one seed falls on each site of $[\![0\,, \lfloor 1/G(\mathbf{a}_\lambda z) \rfloor]\!]$ during the time intervals $[\mathbf{a}_\lambda t_1\,, \mathbf{a}_\lambda t_2]$ and $[\mathbf{a}_\lambda t_3\,, \mathbf{a}_\lambda t_4]$ tends to

$$\begin{cases} 1 & \text{if } t_2 - t_1 > z \text{ and } t_3 - t_4 > z, \\ 0 & \text{if } t_2 - t_1 < z \text{ or } t_3 - t_4 < z. \end{cases}$$

This reinforces our intuition that the random media does not create some substantial time correlations.

The first claim is obvious since for all $0 < s < t$,

$$\mathbb{P}\left[\forall i \in [\![\lfloor a\mathbf{n}_\lambda\rfloor\,,\lfloor b\mathbf{n}_\lambda\rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) - N^S_{\mathbf{a}_\lambda s}(i) > 0\right] \simeq (1 - G(\mathbf{a}_\lambda(t-s))^{(b-a)\mathbf{n}_\lambda}$$
$$\simeq \exp(-(b-a)\mathbf{n}_\lambda G(\mathbf{a}_\lambda(t-s))) \xrightarrow[\lambda\to 0]{} \begin{cases} 1 & \text{if } t-s>1,\\ 0 & \text{if } t-s<1,\end{cases}$$

where we used Lemma A.1. The last claim is also obvious: for all $z \in (0\,,1)$ and all $0 < s < t$,

$$\mathbb{P}\left[\forall i \in [\![0\,,\lfloor 1/G(\mathbf{a}_\lambda z)\rfloor]\!], N^S_{\mathbf{a}_\lambda t}(i) - N^S_{\mathbf{a}_\lambda s}(i) > 0\right] \simeq (1 - G(\mathbf{a}_\lambda(t-s))^{1/G(\mathbf{a}_\lambda z)}$$
$$\simeq \exp(-G(\mathbf{a}_\lambda(t-s)/G(\mathbf{a}_\lambda z))) \xrightarrow[\lambda\to 0]{} \begin{cases} 1 & \text{if } t-s>z,\\ 0 & \text{if } t-s<z,\end{cases}$$

where we used $\mathbf{RV}(\infty)$ in the last step.

Thus, we don't need exact computations (knowing $(\kappa_i)_{i\in\mathbb{Z}}$), since the limit is trivial.

III.3.5. Cluster size distribution

We may easily deduce from Theorem III.3.3 the following estimates on the cluster size distribution.

Corollary III.3.5. *Let ν be a probability distribution on $(0\,,\infty)$ and $(\kappa_i)_{i\in\mathbb{Z}}$ an i.i.d. sequence of random variables with law ν. For each $\lambda \in (0\,,1)$, consider the process $(Z^\lambda_t(x), D^\lambda_t(x))_{t\ge 0, x\in\mathbb{R}}$ associated with the $(\lambda,\nu)-$FFPRM. Consider also the LFF$(\infty)-$process $(Z_t(x), D_t(x), H_t(x))_{t\ge 0,x\in\mathbb{R}}$. Assume* $\mathbf{RV}(\infty)$.

1. *For some $0 < c_1 < c_2$, for all $t \ge \frac{5}{2}$, all $0 < a < b < 1$,*

$$\lim_{\lambda\to 0} \mathbb{P}\left[|C(\eta^\lambda_{\mathbf{a}_\lambda t}, 0)| \in [1/G(\mathbf{a}_\lambda a)\,, 1/G(\mathbf{a}_\lambda b)]\right] = \mathbb{P}\left[Z_t(0) \in [a\,,b]\right] \in [c_1(b-a)\,, c_2(b-a)].$$

2. *For some $0 < c_1 < c_2$ and $0 < \kappa_1 < \kappa_2$, for all $t \ge \frac{3}{2}$, all $B > 0$,*

$$\lim_{\lambda\to 0} \mathbb{P}\left[|C(\eta^\lambda_{\mathbf{a}_\lambda t}, 0)| \ge B\mathbf{n}_\lambda\right] = \mathbb{P}\left[D_t(x) \ge B\right] \in [c_1 e^{-\kappa_2 B}\,, c_2 e^{-\kappa_1 B}].$$

III.4. The case $\beta \in (0\,,\infty)$

III.4.1. Definition of the limit process.

In this case, there are only macroscopic clusters and thus no microscopic fires. This is due to the fact that for $\beta < \infty$, the space scale $\mathbf{n}_\lambda$ is correct for all times. We describe the limit forest fire process by a graphical construction. The limit forest fire process $(Y_t(x))_{t\ge 0,x\in\mathbb{R}}$ will take its values in $\{0,1\}$. In some sense, $Y_t(x) = 0$ means that there is no tree at x at time t.

For $(Y(x))_{x\in\mathbb{R}}$ with values in $\{0,1\}$, we define the occupied component around $x \in \mathbb{R}$ as

$$C(Y,x) := [l(Y,x)\,,r(Y,x)] \tag{III.4.1}$$

where $l(Y,x) = \sup\{y \le x : Y(y) = 0\}$ and $r(Y,x) = \inf\{y \ge x : Y(y) = 0\}$.

If $Y(x) = 0$, then $C(Y,x) = \{x\}$.

We consider a Poisson measure $\pi_M(\mathrm{d}x, \mathrm{d}t)$ on $\mathbb{R} \times [0\,,\infty)$ with intensity measure $\mathrm{d}x\,\mathrm{d}t$, whose marks correspond to matches. We also introduce a Poisson measure $\pi_S(\mathrm{d}x, \mathrm{d}l)$ on $\mathbb{R} \times \mathbb{R}_+$, independent of π_M, with intensity measure

$$\mathrm{d}x \frac{\beta}{\Gamma(\beta+2)} l^{\beta-1}\,\mathrm{d}l.$$

Let us denote by $\{(z_k, l_k) : k \in \mathbb{N}\}$ the marks of π_S. Conditionally on π_S, we consider, for each $k \in \mathbb{N}$, a Poisson process $(M_s(z_k))_{s\ge 0}$ with parameter l_k. Let

$$\mathcal{A} := \{z_k : k \in \mathbb{N}\}.$$

Formally, $(Y_t(x))_{t\ge 0, x\in\mathbb{R}}$ is defined as follows: on all sites $x \in \mathbb{R} \setminus \mathcal{A}$, seeds fall continuously while for all $k \in \mathbb{N}$, seeds fall on z_k according to $(M_s(z_k))_{s\ge 0}$.

At time 0^+, all sites are occupied except those of $\mathcal{A}$. When a match falls on some site $x_0 \in \mathbb{R}$ at some time t_0, it destroys the corresponding connected component C (necessarily delimited by two elements of $\mathcal{A}$). All the sites of $C \setminus \mathcal{A}$ are immediately occupied again (at time t_0^+) while the sites of $C \cap \mathcal{A}$ wait for the next seed (first jump of their Poisson process after t_0) to become occupied again.

For convenience, we slightly change these rules: we simply set $Y_t(x) = 1$ for all $t \ge 0$ and all $x \in C \setminus \mathcal{A}$: in other words, for sites where seeds fall continuously, we do not formalize the instantaneous changes from 1 to 0 to 1.

A rigorous construction is not hard to handle. Fix $T > 0$. First, we easily find a.s. a sequence $(\chi_i)_{i\in\mathbb{Z}} \subset \mathcal{A}$ satisfying the conditions that

$$\chi_i \le \chi_{i+1}, \quad \chi_0 \le 0 \le \chi_1, \quad \lim_{i\to-\infty} \chi_i = -\infty, \quad \lim_{i\to\infty} \chi_i = \infty, \quad M_T(\chi_i) = 0.$$

We set $Y_t(\chi_i) = 0$ for all $i \in \mathbb{Z}$ and all $t \in [0\,,T]$ and handle the construction separately on each $(\chi_i\,,\chi_{i+1})$. Let thus i be fixed. Denote by $(\alpha_l^i, \rho_l^i)_{l=1,\ldots L^i}$ the marks of π_M in $[0\,,T] \times (\chi_i\,,\chi_{i+1})$ ordered chronologically.

For $t \in [0\,,\rho_1^i)$, we put $Y_t(x) = 1$ for all $x \in (\chi_i\,,\chi_{i+1}) \setminus \mathcal{A}$ and

$$Y_t(x) = \min(M_t(x), 1) \quad \text{for all} \quad x \in (\chi_i\,,\chi_{i+1}) \cap \mathcal{A}.$$

This allows us to define the connected component of α_1^i at time ρ_1^i-. We thus set

$$I_1^i = C(Y_{\rho_1^i-}, \alpha_1^i)$$

and

$$Y_{\rho_1^i}(x) = \begin{cases} 1 & \text{for all } x \in (\chi_i\,,\chi_{i+1}) \setminus \mathcal{A}, \\ Y_{\rho_1^i-}(x) & \text{for all } x \in \mathcal{A} \setminus I_1^i, \\ 0 & \text{for all } x \in \mathcal{A} \cap I_1^i. \end{cases}$$

Next, for all $t \in [\rho_1^i, \rho_2^i)$, we put $Y_t(x) = 1$ for all $x \in (\chi_i, \chi_{i+1}) \setminus \mathcal{A}$ and

$$Y_t(x) = \min(Y_{\rho_1^i}(x) + M_t(x) - M_{\rho_1^i}(x), 1) \quad \text{for all} \quad x \in (\chi_i, \chi_{i+1}) \cap \mathcal{A}.$$

And so on.

A typical path of the LFF(β)−process is drawn on Figure III.2.

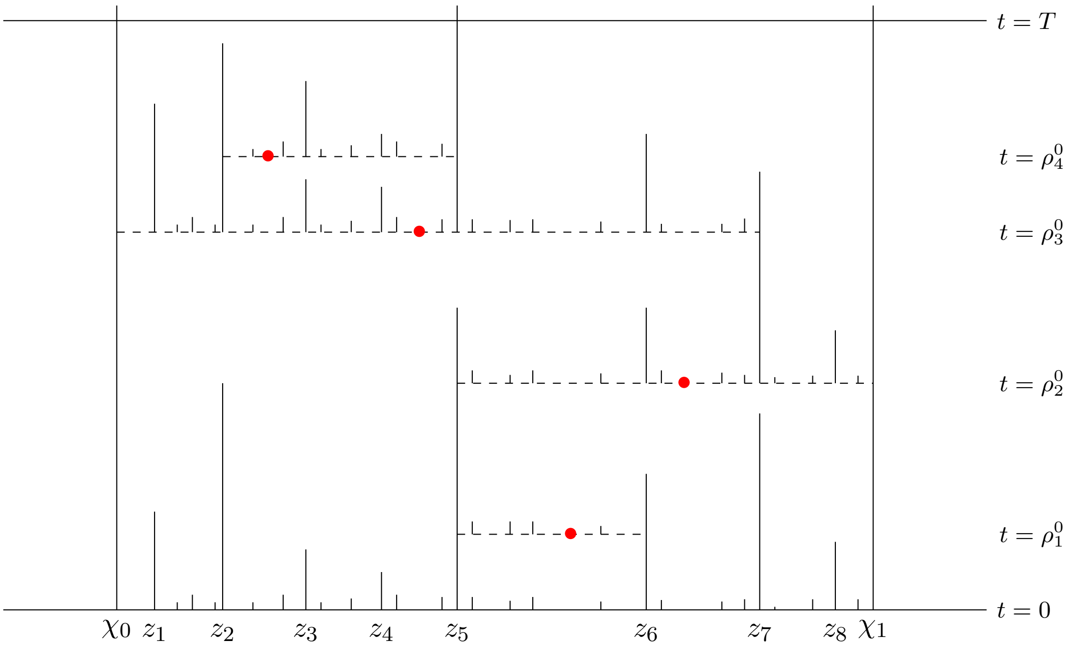

Figure III.2.: LFF(β)−process with $\beta \in (0, \infty)$.

The plain segments represent vacant sites and occupied clusters are delimited by these segments. The marks of π_M (matches) are represented as bullets.
No seed falls on χ_0 nor on χ_1 (which are marks of π_S) during $[0, T]$. Let $(z_k, l_k)_{k\in\mathbb{N}}$ be the marks of π_S in $(\chi_0, \chi_1) \times \mathbb{R}_+$. Observe that, for all $\varepsilon > 0$, $\{k \in \mathbb{N} : l_k \geq \varepsilon\}$ is an infinite countable set while $\{k \in \mathbb{N} : l_k < \varepsilon\}$ is a finite set. Thus we cannot draw *exactly* the process on any finite interval (χ_i, χ_{i+1}). Seeds fall continuously except on z_k, for all $k \in \mathbb{N}$, where seeds fall according to a Poisson process with parameter l_k.
We fix some small $\varepsilon > 0$ and we call $(z_k, l_k)_{k=1,\dots,8}$ the marks of π_S in $(\chi_0, \chi_1) \times \mathbb{R}_+$ with $l_k < \varepsilon$.
When the first match falls at time ρ_1^0, no seed has fallen on z_5 and z_6 while at least one seed has fallen on each other site which belong to (z_5, z_6). Thus, this match destroys the zone (z_5, z_6).
When the second match falls at time ρ_2^0, no seed has fallen on z_5 and χ_1 while all the other sites contained in the zone (z_5, χ_1) are occupied. Thus, the match destroys the zone (z_5, χ_1). Seeds fall on z_6, z_7 and z_8 according to Poisson processes with respective parameter l_6, l_7 and l_8.
For example, the height of the two plain segments above z_7 are two independent exponential random variables with parameter l_7.

Proposition III.4.1. *Let π_M, π_S be two independent Poisson measures on $\mathbb{R} \times [0, \infty)$ and $\mathbb{R} \times [0, \infty)$ with intensity measures $\mathrm{d}x\,\mathrm{d}t$ and $\mathrm{d}x(\beta/\Gamma(\beta+2))l^{\beta-1}\,\mathrm{d}l$. There a.s. exists* *a unique LFF(β)−process $(Y_t(x))_{t\geq 0,x\in\mathbb{R}}$. It can be simulated exactly on any finite box $[-n\,,n]\times[0\,,T]$. For each $t\geq 0$ and $x\in\mathbb{R}$, we will denote by $D_t(x)=C(Y_t,x)$, recall* (III.4.1).

III.4.2. The convergence result

We now state our expected result in the case $\beta\in(0\,,\infty)$. We use Subsection III.1.3. A heuristic proof will be given Subsection.

Theorem III.4.2. *Let ν be a probability distribution on $(0\,,\infty)$ satisfying* $\mathbf{RV}(\beta)$, *for some $\beta\in(0\,,\infty)$. Let $(\kappa_i)_{i\in\mathbb{Z}}$ be an i.i.d. sequence of random variables with law ν. Consider, for each $\lambda\in(0\,,1]$, the process $(D_t^\lambda(x))_{t\geq 0,x\in\mathbb{R}}$ associated with the (λ,ν)−FFPRM. Consider also the LFF(β)−process $(Y_t(x))_{t\geq 0,x\in\mathbb{R}}$ and the associated $(D_t(x))_{t\geq 0,x\in\mathbb{R}}$.*

1. *For any $T>0$, any finite subset $\{x_1,\ldots,x_p\}\subset\mathbb{R}$,*

$$(D_t^\lambda(x_i))_{t\in[0,T],i=1,\ldots,p}$$

   goes in law to $(D_t(x_i))_{t\in[0,T],i=1,\ldots,p}$ in $\mathbb{D}([0\,,T],\mathcal{I})^p$, as λ tends to 0. Here $\mathbb{D}([0\,,T],\mathcal{I})$ is endowed with the distance $\boldsymbol{\delta}_T$.

2. *For any finite subset $\{(x_1,t_1),\ldots,(x_p,t_p)\}\subset\mathbb{R}\times(0\,,\infty)$, $(D_{t_i}^\lambda(x_i))_{i=1,\ldots,p}$ goes in law to $(D_{t_i}(x_i))_{t\in[0,T],i=1,\ldots,p}$ in $\mathcal{I}^p$, as λ tends to 0, $\mathcal{I}$ being endowed with $\boldsymbol{\delta}$.*

Observe that the random media is still present in the limit process through the Poisson measure π_S.

Example 3. Consider the case where ν is a Gamma distribution with parameter $\beta\in(0\,,\infty)$. It is easy to show that G satisfies $\mathbf{RV}(\beta)$ and that (see Appendix A, Example 3),

$$\mathbf{a}_\lambda \underset{\lambda\to 0}{\sim}\left(\frac{\beta+1}{\lambda}\right)^{1/(\beta+1)}.$$

III.4.3. Heuristic arguments

Recall that $\mathbf{a}_\lambda$ and $\mathbf{n}_\lambda$ are defined in (III.2.2) and (III.2.5).

1. A convincing easy computation. Here we show that for all $t>0$, neglecting fires, denoting by $R_t^\lambda=\inf\left\{i\geq 0:\eta^\lambda_{\mathbf{a}_\lambda t}(i)=0\right\}$ and by $R_t=\inf\{x>0:Y_t(x)=0\}$, we have

$$\frac{R_t^\lambda}{\mathbf{n}_\lambda}\xrightarrow[\lambda\to 0]{\mathcal{L}} R_t.$$

This suggests that the intensity of π_S should be the right one.

Since we neglect fires and since each site is vacant at time $\mathbf{a}_\lambda t$ with probability $\mathbb{E}\left[e^{-\mathbf{a}_\lambda \kappa_i t}\right] = G(\mathbf{a}_\lambda t)$, R_t^λ is nothing but a geometric random variable with parameters $G(\mathbf{a}_\lambda t)$. Using Lemma A.1, we have

$$\mathbf{n}_\lambda G(\mathbf{a}_\lambda t) \underset{\lambda\to 0}{\longrightarrow} \frac{1}{(\beta+1)t^\beta}$$

We easily deduce that $R_t^\lambda/\mathbf{n}_\lambda$ converges in law to an exponential random variable with parameter $1/((\beta+1)t^\beta)$.

The link with the LFF$(\beta)-$process is simple since, neglecting fires, the random variable R_t follows an exponential law with parameter

$$\frac{\beta}{\Gamma(\beta+2)}\int_0^\infty \int_t^\infty l e^{-lr} l^{\beta-1}\,\mathrm{d}r\,\mathrm{d}l = \frac{\beta}{\Gamma(\beta+2)}\int_0^\infty l^{\beta-1}e^{-tl}\,\mathrm{d}l = \frac{1}{(\beta+1)t^\beta}.$$

2. The Poisson measure. Here we want to explain that π_S will be obtained as the limit of

$$\pi_S^\lambda = \sum_{i\in\mathbb{Z}} \delta_{(i/\mathbf{n}_\lambda, \mathbf{a}_\lambda \kappa_i)}.$$

Let us *e.g.* show that, for all $a<b$ and all $K>0$,

$$\left|Z_{a,b,K}^\lambda\right| \underset{\lambda\to 0}{\overset{\mathcal{L}}{\longrightarrow}} \mathcal{P}\left(\frac{(b-a)K^\beta}{\Gamma(\beta+2)}\right) \tag{III.4.2}$$

where $Z_{a,b,K}^\lambda := \pi_S^\lambda([a\,,b]\times[0\,,K])$ (observe that $\int_0^K \beta l^{\beta-1}\,\mathrm{d}l = K^\beta$).

Since $G(t) = L(t)/t^\beta$, where $L(t) = t^\beta G(t)$ is a slowly varying function, we can argue that, using Theorem 15.3 p.30 in [Kor04],

$$\nu\left((0\,,\varepsilon)\right) \underset{\varepsilon\to 0}{\sim} \frac{L(1/\varepsilon)}{\Gamma(\beta+1)}\varepsilon^\beta = \frac{G(1/\varepsilon)}{\Gamma(\beta+1)}$$

whence, using Lemma A.1,

$$(\lfloor b\mathbf{n}_\lambda\rfloor - \lfloor a\mathbf{n}_\lambda\rfloor + 1)\times\nu((0\,,K/\mathbf{a}_\lambda)) \underset{\lambda\to 0}{\sim} \frac{(b-a)}{\Gamma(\beta+1)}\times\mathbf{n}_\lambda\times G(\mathbf{a}_\lambda/K) \underset{\lambda\to 0}{\longrightarrow} \frac{(b-a)K^\beta}{\Gamma(\beta+1)(\beta+1)}.$$

The conclusion follows easily because $\left|Z_{a,b,K}^\lambda\right|$ has a binomial distribution with parameters $\lfloor b\mathbf{n}_\lambda\rfloor - \lfloor a\mathbf{n}_\lambda\rfloor + 1$ and $\nu((0\,,K/\mathbf{a}_\lambda))$.

3. Occupation of vacant zones: first argument. Here we claim that for all sites not concerned by π_S^λ, seeds fall almost continuously as for the limit process. More precisely, we will verify in Lemma A.2 that for all $0<s<t$ and all $a<b$,

$$\lim_{K\to\infty}\inf_{\lambda\in(0,1)} \mathbb{P}\left[\forall i \in [\![\lfloor a\mathbf{n}_\lambda\rfloor\,,\lfloor b\mathbf{n}_\lambda\rfloor]\!] \setminus Z_{a,b,K}^\lambda, N_{\mathbf{a}_\lambda t}^S(i) - N_{\mathbf{a}_\lambda s}^S(i) > 0\right] = 1. \tag{III.4.3}$$

4. Occupation of vacant zones: second argument. Consider (x, l) such that $\pi_S^\lambda(\{(x, l)\}) \approx \pi_S(\{x, l\}) = 1$. Then on $i := \lfloor x\mathbf{n}_\lambda \rfloor$, seeds fall according to a Poisson process with rate $\kappa_i = l/\mathbf{a}_\lambda$, and thus with rate $\mathbf{a}_\lambda \kappa_i = l$ after acceleration of time by $\mathbf{a}_\lambda$.

In the limit process, seeds fall on x according to a Poisson process with rate l: this is very similar.

5. Conclusion. We have seen in points 4 and 5 that in the discrete process, seeds fall almost continuously on sites not concerned by π_S^λ (as in the limit process) and according to Poisson processes with the good rate on sites concerned by π_S^λ (as in the limit process). Clearly, fires have the same effect on both processes. Thus, the two processes should behave similarly.

III.4.4. Cluster size distribution

We aim here to estimate the law of the occupied cluster around 0. We expect the following behavior.

Lemma III.4.3. *Let $\beta \in (0, \infty)$. Consider the LFF(β)$-$process $(Y_t(x))_{t\geq 0, x\in\mathbb{R}}$ and the associated $(D_t(x))_{t\geq 0, x\in\mathbb{R}}$. Consider a probability distribution ν on $(0, \infty)$ as well as, for each $\lambda \in (0, 1]$, a $(\lambda, \nu)-$FFPRM $(\eta_t^\lambda(i))_{t\geq 0, i\in\mathbb{Z}}$. Assume $\mathbf{RV}(\beta)$. There are some constants $0 < c_1 < c_2$ and $0 < \kappa_1 < \kappa_2$ such that for all $t \geq 1$ and all $B > 0$,*

$$\lim_{\lambda\to 0} \mathbb{P}\left[\left|C(\eta_{\mathbf{a}_\lambda t}^\lambda, 0)\right| \geq B\mathbf{n}_\lambda\right] = \mathbb{P}\left[|D_t(0)| \geq B\right] \in [c_1 e^{-\kappa_2 B}, c_2 e^{-\kappa_1 B}].$$

III.5. The case $\beta = 0$

III.5.1. Definition of the limit process

In this case, the limiting process is trivial: we consider a Poisson measure π_S on $\mathbb{R}$ with intensity measure $\mathrm{d}x$ and we put, for all $t \geq 0$, all $x \in \mathbb{R}$,

$$Y_t(x) = \mathbf{1}_{\{\pi_S(x)=0\}}. \tag{III.5.1}$$

Denote by $\{\chi_i\}_{i\in\mathbb{Z}}$ the marks of π_S with the convention that $\cdots < \chi_{-1} < 0 < \chi_0 < \dots$. Then for all $t \geq 0$, all $i \in \mathbb{Z}$, recalling (III.4.1),

$$C(Y_t, x) = [\chi_i, \chi_{i+1}] \tag{III.5.2}$$

for all $x \in (\chi_i, \chi_{i+1})$ and $C(Y_t, \chi_i) = \{\chi_i\}$.

Proposition III.5.1. *Let π_S be a Poisson measure on $\mathbb{R}$ with intensity measure $\mathrm{d}x$. There obviously a.s. exists a unique LFF(0)$-$process $(Y_t(x))_{t\geq 0, x\in\mathbb{R}}$. It can be simulated exactly on any finite box $[-n, n] \times [0, \infty]$. For each $t \geq 0$ and $x \in \mathbb{R}$, we will denote $D_t(x) = C(Y_t, x)$ the occupied cluster around x, see (III.4.1).*

We do not see fires at the limit but we should keep in mind that when a match falls, it destroys a zone which becomes immediately occupied, because seeds fall continuously on almost every sites. This zone is delimited by sites where seeds never falls. A typical path of the LFF(0)$-$process is drawn on Figure III.3.

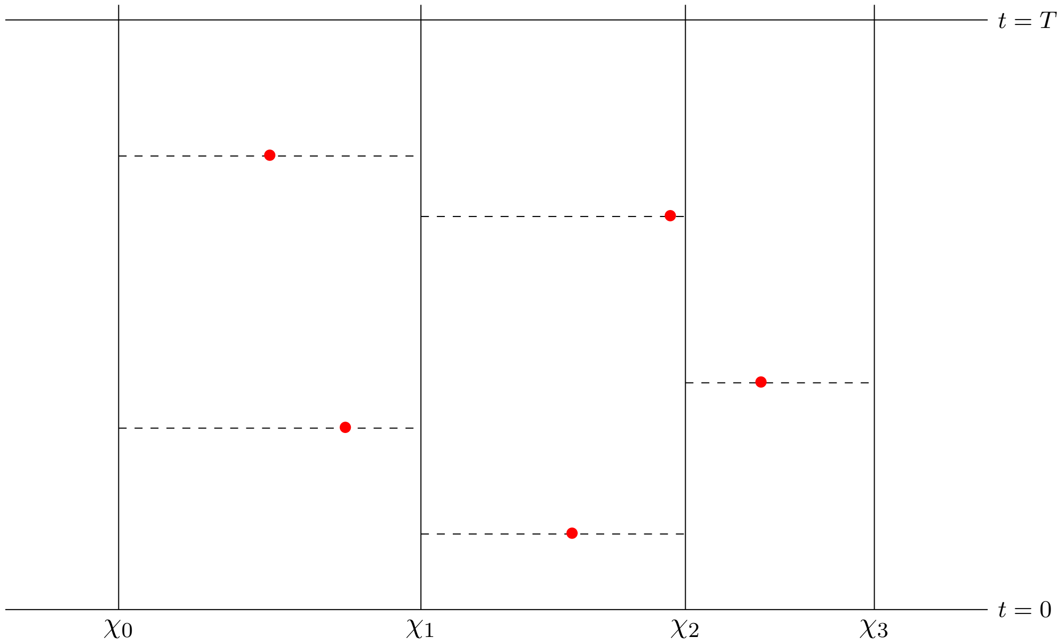

Figure III.3.: LFF(0)−process.

The marks of π_M (matches) are represented as bullets. We draw a plain vertical segment above each mark of π_S. For all times, the occupied clusters are delimited by these vertical segments. In some sense, fires have an instantaneous effect, represented as dotted horizontal segments.

III.5.2. The convergence result

We now state our expected result in the case $\beta = 0$. We use Subsection III.1.3. A heuristic proof will be given in next Subsection.

Theorem III.5.2. *Let ν be a probability distribution on $(0, \infty)$ satisfying* $\mathbf{RV}(0)$*. Consider $(\kappa_i)_{i\in\mathbb{Z}}$ an i.i.d. sequence of random variables with law ν. Consider, for each $\lambda \in (0, 1]$, the process $(D_t^\lambda(x))_{t\geq 0, x\in\mathbb{R}}$ associated with the $(\lambda, \nu)-$FFPRM. Consider also the LFF$(0)-$process $(Y_t(x))_{t\geq 0, x\in\mathbb{R}}$ and the associated $(D_t(x))_{t\geq 0, x\in\mathbb{R}}$.*

1. *For any $T > 0$, any finite subset $\{x_1, \dots, x_p\} \subset \mathbb{R}$, $(D_t^\lambda(x_i))_{t\in[0,T], i=1,\dots,p}$ goes in law to $(D_t(x_i))_{t\in[0,T], i=1,\dots,p}$ in $\mathbb{D}([0, T], \mathcal{I})^p$, as λ tends to 0. Here $\mathbb{D}([0, T], \mathcal{I})$ is endowed with the distance $\boldsymbol{\delta}_T$.*

2. *For any finite subset $\{(x_1, t_1), \dots, (x_p, t_p)\} \subset \mathbb{R} \times (0, \infty)$, $(D_{t_i}^\lambda(x_i))_{i=1,\dots,p}$ goes in law to $(D_{t_i}(x_i))_{t\in[0,T], i=1,\dots,p}$ in $\mathcal{I}^p$, as λ tends to 0, $\mathcal{I}$ being endowed with $\boldsymbol{\delta}$.*

III.5.3. Heuristic arguments

Here we give a heuristic proof of Theorem III.5.2.

1. Convergence of seed process. For $\beta > 0$, we define the measure

$$q_\beta(\mathrm{d}l) = \frac{\beta l^{\beta-1}}{\Gamma(\beta+2)} \mathbf{1}_{\{l \geq 0\}} \,\mathrm{d}l.$$

Loosely speaking , the measure q_β converges, as $\beta \to 0$, to the Dirac mass at 0 in the sense where, for all $L > 0$,

$$q_\beta([0\,,L]) = \frac{\beta}{\Gamma(\beta+2)} \int_0^L l^{\beta-1}\,\mathrm{d}l = \frac{L^\beta}{\Gamma(\beta+2)} \xrightarrow[\beta\to 0]{} 1$$

while, for all $0 < A < B$,

$$q_\beta([A\,,B]) = \frac{\beta}{\Gamma(\beta+2)} \int_A^B l^{\beta-1}\,\mathrm{d}l = \frac{1}{\Gamma(\beta+2)}(B^\beta - A^\beta) \xrightarrow[\beta\to 0]{} 0.$$

Thus, when β tends to 0, the LFF$(\beta)-$process tends (in a weak sense) to the LFF$(0)-$process, where there are only two kinds of sites: *slow sites* where the first seed never falls (*i.e.* seeds fall according to a Poisson process with parameter 0) and *fast sites* where seeds fall continuously.

2. The Poisson measure. For $K > 0$ and $a < b$, let us denote by

$$Z^\lambda_{a,b,K} \coloneqq \{i \in [\![\lfloor a\mathbf{n}_\lambda \rfloor\,, \lfloor b\mathbf{n}_\lambda \rfloor]\!] : \kappa_i \leq K/\mathbf{a}_\lambda\}$$

the (random) set of sites which have an *abnormally small parameter*.

Using similar arguments as in point 2 in the heuristic proof in Subsection III.4.3, it is easy to show that

$$\left|Z^\lambda_{a,b,K}\right| \xrightarrow[\lambda\to 0]{\mathcal{L}} \mathcal{P}((b-a)).$$

This last quantity does not depend on $K > 0$.

3. Occupation of vacant zones. Hence roughly, for $\lambda > 0$ very small, $Z^\lambda_{-\infty,\infty} \simeq Z^\lambda_{-\infty,\infty,K}$ (roughly, for all K). As a consequence, there are only two types of sites: sites of $Z^\lambda_{-\infty,\infty}$, for which $\mathbf{a}_\lambda \kappa_i \ll 1$, on which the first seed will never fall (in our time scale), and sites of $\mathbb{Z} \setminus Z^\lambda_{-\infty,\infty}$, for which $\mathbf{a}_\lambda \kappa_i \gg 1$, on which seeds will fall almost continuously (in our time scale). Slow sites are located, roughly, according to a Poisson, measure with intensity 1 on $\mathbb{R}$ (after rescaling of $\mathbb{Z}$ by $\mathbf{n}_\lambda$).

4. Conclusion. Comparing the arguments above, we hope that, if G satisfies $\mathbf{RV}(0)$, when λ tends to 0, the $(\lambda,\nu)-$FFPRM converges to the LFF$(0)-$process.

III.5.4. Cluster size distribution

The LFF(0)-process is very simple and the following is obvious.

Corollary III.5.3. *Let ν be a probability distribution on $(0\,,\infty)$ and $(\kappa_i)_{i\in\mathbb{Z}}$ an i.i.d. sequence of random variables with respect to the law ν. Consider, for each $\lambda \in (0\,,1]$, the process $(D_t^\lambda(x))_{t\geq 0, x\in\mathbb{R}}$ associated with the $(\lambda,\nu)-$FFPRM. Consider also the LFF$(0)-$process $(Y_t(x))_{t\geq 0, x\in\mathbb{R}}$ and the associated $(D_t(x))_{t\geq 0, x\in\mathbb{R}}$. Assume* $\mathbf{RV}(0)$*. Then, for $t > 0$ and $B > 0$,*

$$\lim_{\lambda\to 0} \mathbb{P}\left[\left|C(\eta^\lambda_{\mathbf{a}_\lambda t}, 0)\right| \geq B\mathbf{n}_\lambda\right] = \mathbb{P}\left[|D_t(0)| \geq B\right] = \int_B^\infty x e^{-x}\,\mathrm{d}x = (B+1)e^{-B}.$$

A. Appendix.

In this Appendix, we first prove (III.2.7) and (III.4.3). We next prove the existence of a function $\mathbf{m}_\lambda$ satisfying (III.2.8). Finally, we study the three examples encountered in the paper.

A.1. Some well known results about regularly varying functions

We first prove (III.2.7).

Lemma A.1. *Let ν be a probability distribution on $(0\,,\infty)$ with Laplace transform G. Recall (III.2.2), (III.2.3) and (III.2.5). If G satisfies $\mathbf{RV}(\beta)$, for some $\beta \in [0\,,\infty)\cup\{\infty\}$, then for all $t \neq 1$,*

$$\frac{1}{\mathbf{n}_\lambda G(\mathbf{a}_\lambda t)} \xrightarrow[\lambda\to 0]{} \begin{cases} (\beta+1)t^\beta & \text{if } \beta \in [0\,,\infty), \\ 0 & \text{if } \beta = \infty \text{ and } t < 1, \\ 1 & \text{if } \beta = \infty \text{ and } t > 1. \end{cases}$$

Furthermore, if G satisfies $\mathbf{RV}(\infty)$, then $\mathbf{n}_\lambda G(\mathbf{a}_\lambda)$ tends to 0 when λ tends to 0.

Proof. Let us first assume that G satisfies $\mathbf{RV}(\beta)$ for some $\beta \in [0\,,\infty)$. Thus, $1/G$ has a representation

$$\frac{1}{G(t)} = t^\beta L(t)$$

where L is some slowly varying function. By Karamata's Theorem ([Kor04], Proposition 5.1 p186), since $\mathbf{a}_\lambda \to \infty$, we can argue that

$$\int_0^{\mathbf{a}_\lambda} \frac{1}{G(s)}\,\mathrm{d}s = \int_0^{\mathbf{a}_\lambda} t^\beta L(s)\,\mathrm{d}s \sim L(\mathbf{a}_\lambda)\int_0^{\mathbf{a}_\lambda} t^\beta\,\mathrm{d}s = \frac{\mathbf{a}_\lambda^{\beta+1}}{\beta+1}L(\mathbf{a}_\lambda) = \frac{1}{\beta+1}\frac{\mathbf{a}_\lambda}{G(\mathbf{a}_\lambda)}.$$

Recalling (III.2.2) and (III.2.5) and using $\mathbf{RV}(\beta)$, we easily deduce (III.2.7) for $\beta \in [0\,,\infty)$.

We next assume that G satisfies $\mathbf{RV}(\infty)$. Recall (III.2.2) and (III.2.5) and observe that

$$\mathbf{n}_\lambda G(\mathbf{a}_\lambda t) \simeq \frac{G(\mathbf{a}_\lambda t)}{\lambda \mathbf{a}_\lambda} = \frac{1}{\mathbf{a}_\lambda}\int_0^{\mathbf{a}_\lambda} \frac{G(\mathbf{a}_\lambda t)}{G(s)}\,\mathrm{d}s = \int_0^1 \frac{G(\mathbf{a}_\lambda t)}{G(\mathbf{a}_\lambda s)}\,\mathrm{d}s.$$

This last quantity obviously tends to 0 as $\lambda \to 0$ when $t \geq 1$ (using $\mathbf{RV}(\infty)$, (III.2.3) and the dominated convergence theorem) and tends to ∞ when $t < 1$ because then

$$\int_0^1 \frac{G(\mathbf{a}_\lambda t)}{G(\mathbf{a}_\lambda s)}\,\mathrm{d}s \geq \int_{(t+1)/2}^1 \frac{G(\mathbf{a}_\lambda t)}{G(\mathbf{a}_\lambda s)}\,\mathrm{d}s \geq \frac{1-t}{2}\frac{G(\mathbf{a}_\lambda t)}{G(\mathbf{a}_\lambda (t+1)/2)} \xrightarrow[\lambda\to 0]{} \infty. \qquad \square$$

We next prove (III.4.3).

Lemma A.2. *Consider a probability distribution ν satisfying* $\mathbf{RV}(\beta)$ *for some* $\beta \in (0\,,\infty)$. *Let* $(\kappa_i)_{i\in\mathbb{Z}}$ *be an i.i.d. sequence of random variables with law* ν. *For all* $\lambda \in (0\,,1)$, *all* $a < b$ *and all* $K > 0$, *consider the random set*

$$Z^{\lambda}_{a,b,K} = \{i \in [\![\lfloor a\mathbf{n}_\lambda\rfloor\,,\lfloor b\mathbf{n}_\lambda\rfloor]\!] : \mathbf{a}_\lambda\kappa_i \le K\}\,.$$

Then, for all $0 \le s < t$, *there holds that*

$$\lim_{K\to\infty}\inf_{\lambda\in(0,1)}\mathbb{P}\left[\forall i \in [\![\lfloor a\mathbf{n}_\lambda\rfloor\,,\lfloor b\mathbf{n}_\lambda\rfloor]\!]\setminus Z^{\lambda}_{a,b,K}, N^S_{\mathbf{a}_\lambda t}(i) - N^S_{\mathbf{a}_\lambda s}(i) > 0\right] = 1.$$

Proof. By time stationarity, we assume that $s = 0$. First, for all $\lambda \in (0\,,1)$, by space stationarity

$$\begin{aligned}\mathbb{P}\left[\exists i \in [\![\lfloor a\mathbf{n}_\lambda\rfloor\,,\lfloor b\mathbf{n}_\lambda\rfloor]\!]\setminus Z^{\lambda}_{a,b,K}, N^S_{\mathbf{a}_\lambda t}(i) = 0\right] &\le (b-a+1)\mathbf{n}_\lambda\mathbb{P}\left[\kappa_0 > K/\mathbf{a}_\lambda,\ N^S_{\mathbf{a}_\lambda t}(0) = 0\right]\\ &= (b-a+1)\mathbf{n}_\lambda\int_{K/\mathbf{a}_\lambda}^{\infty} e^{-\mathbf{a}_\lambda tr}\nu(\mathrm{d}r). \quad (A.1)\end{aligned}$$

But

$$\begin{aligned}\int_{K/\mathbf{a}_\lambda}^{\infty} e^{-\mathbf{a}_\lambda tr}\nu(\mathrm{d}r) &= \left[\nu((0\,,r))e^{-\mathbf{a}_\lambda tr}\right]_{r=K/\mathbf{a}_\lambda}^{\infty} + \mathbf{a}_\lambda t\int_{K/\mathbf{a}_\lambda}^{\infty}\nu((0\,,r))e^{-\mathbf{a}_\lambda tr}\,\mathrm{d}r\\ &= -\nu((0\,,K/\mathbf{a}_\lambda))e^{-Kt} + \int_{Kt}^{\infty} e^{-x}\nu((0\,,x/(\mathbf{a}_\lambda t)))\,\mathrm{d}x\\ &\le \int_{Kt}^{\infty} e^{-x}\nu((0\,,x/(\mathbf{a}_\lambda t)))\,\mathrm{d}x\end{aligned}$$

whence

$$0 \le \mathbf{n}_\lambda\int_{K/\mathbf{a}_\lambda}^{\infty} e^{-\mathbf{a}_\lambda tr}\nu(\mathrm{d}r) \le \mathbf{n}_\lambda\int_{Kt}^{\infty} e^{-x}\nu((0\,,x/(\mathbf{a}_\lambda t)))\,\mathrm{d}x. \quad (A.2)$$

On the one hand, since G satisfies $\mathbf{RV}(\beta)$, using Theorem 15.3 p.30 in [Kor04], which ensures us that

$$\nu\left((0\,,\varepsilon)\right) \underset{\varepsilon\to 0}{\sim} \frac{G(1/\varepsilon)}{\Gamma(\beta+1)},$$

we deduce that there is $C > 0$ such that for all $x \in (0\,,1]$,

$$\nu((0\,,x)) \le C \times G(1/x).$$

On the other hand, for all $x \ge 1$, we easily have

$$G(1/x) \ge G(1) \ge G(1) \times \nu((0\,,x)).$$

Hence, there is $C > 0$ such that, for all $x > 0$, $\nu((0\,,x)) \le C \times G(1/x)$ and we may write

$$\mathbf{n}_\lambda\int_{Kt}^{\infty} e^{-x}\nu((0\,,x/(\mathbf{a}_\lambda t)))\,\mathrm{d}x \le \mathbf{n}_\lambda \times C \times \int_{Kt}^{\infty} e^{-x}G(\mathbf{a}_\lambda t/x)\,\mathrm{d}x. \quad (A.3)$$

Recalling (III.2.5) and using Karamata's Theorem ([Kor04], Proposition 5.1 p. 186), we deduce that

$$\mathbf{n}_\lambda \sim \frac{1}{\lambda \mathbf{a}_\lambda} = \frac{1}{\mathbf{a}_\lambda} \times \int_0^{\mathbf{a}_\lambda} \frac{1}{G(s)} \, \mathrm{d}s \underset{\lambda \to 0}{\sim} \frac{1}{(\beta+1) G(\mathbf{a}_\lambda)}.$$

Hence there is $C > 0$ such that for all $\lambda \in (0\,,1)$,

$$\mathbf{n}_\lambda \int_{Kt}^{\infty} e^{-x} G(\mathbf{a}_\lambda t/x) \, \mathrm{d}x \le C \int_{Kt}^{\infty} e^{-x} \frac{G(\mathbf{a}_\lambda t/x)}{G(\mathbf{a}_\lambda)} \, \mathrm{d}x. \tag{A.4}$$

An immediate consequence of the representation theorem of slowly varying function ([Kor04], Theorem 2.2 p. 180) ensures us that for any $\gamma > 0$, there is $b = b(\gamma) > 0$ such that for all $0 \le v \le u < \infty$,

$$\frac{u^\beta}{v^\beta} \frac{G(u)}{G(v)} \ge b \left(\frac{u+1}{v+1} \right)^{-\gamma},$$

which implies that

$$\frac{G(v)}{G(u)} \le \frac{1}{b} \left(\frac{u}{v} \right)^{\beta} \left(\frac{u}{v} + 1 \right)^{\gamma}.$$

Hence there is $C > 0$ such that for all $\lambda \in (0\,,1)$, all $K \ge 1$ and all $x \ge Kt$,

$$\frac{G(\mathbf{a}_\lambda t/x)}{G(\mathbf{a}_\lambda)} \le C \left(\frac{x}{t} \right)^{\beta+1}. \tag{A.5}$$

Gathering (A.1), (A.2), (A.3), (A.4) and (A.5), we deduce that there is $C > 0$ such that for all $\lambda \in (0\,,1)$ and all $K \ge 1$,

$$\mathbb{P}\left[\exists i \in [\![\lfloor a \mathbf{n}_\lambda \rfloor \,, \lfloor b \mathbf{n}_\lambda \rfloor]\!] \setminus Z^{\lambda}_{a,b,K}, N^{S}_{\mathbf{a}_\lambda t}(i) = 0 \right] \le C(b-a) \int_{Kt}^{\infty} \left(\frac{x}{t} \right)^{\beta+1} e^{-x} \, \mathrm{d}x.$$

Taking the supremum over $\lambda \in (0\,,1)$ and letting K tend to ∞, we deduce the claim. □

A.1. Existence of a function $\mathbf{m}_\lambda$

Lemma A.3. *Let ν a probability distribution on $(0\,,\infty)$. Assume* $\mathbf{RV}(\infty)$. *There exists a function $\mathbf{m}_\lambda \colon (0\,,1] \to \mathbb{N}$ satisfying* (III.2.8).

Proof. By Lemma A.1, there holds that for any $n \ge 1$,

$$\lim_{\lambda \to 0} \frac{\lambda \mathbf{a}_\lambda}{G(\mathbf{a}_\lambda (1 - 1/n))} = 0 \text{ and } \lim_{\lambda \to 0} \frac{G(\mathbf{a}_\lambda)}{\lambda \mathbf{a}_\lambda} = 0.$$

Thus there exists $\lambda_n \in (0\,,1]$ such that for all $\lambda \in (0\,,\lambda_n)$,

$$\frac{\lambda \mathbf{a}_\lambda}{G(\mathbf{a}_\lambda (1 - 1/n))} \le 1/n \text{ and } \frac{G(\mathbf{a}_\lambda)}{\lambda \mathbf{a}_\lambda} \le 1/(4n).$$

We ay of course choose the sequence $(\lambda_n)_{n\geq 1}$ decreasing to 0. Then we define $\varepsilon_\lambda\colon (0\,,1] \to (0\,,1]$ by setting, for all $n \geq 1$, $\varepsilon_\lambda = 1/n$ for $\lambda \in (\lambda_{n+1}\,,\lambda_n]$. There holds $\lim_{\lambda\to 0}\varepsilon_\lambda = 0$. Finally, we put

$$\mathbf{m}_\lambda = \left\lfloor \frac{1}{G(\mathbf{a}_\lambda(1-\varepsilon_\lambda))} \right\rfloor.$$

This function is obviously non-increasing and satisfies, for all $n \geq 1$, all $\lambda \in (\lambda_{n+1}\,,\lambda_n)$,

$$\frac{\mathbf{m}_\lambda}{\mathbf{n}_\lambda} \simeq \frac{\lambda \mathbf{a}_\lambda}{G(\mathbf{a}_\lambda(1-\varepsilon_\lambda))} = \frac{\lambda \mathbf{a}_\lambda}{G(\mathbf{a}_\lambda(1-1/n))} \leq \frac{1}{n}$$

and

$$(2\mathbf{m}_\lambda + 1)G(\mathbf{a}_\lambda) \leq 3\frac{G(\mathbf{a}_\lambda)}{G(\mathbf{a}_\lambda(1-\varepsilon_\lambda))} = 3\frac{G(\mathbf{a}_\lambda)}{\lambda \mathbf{a}_\lambda}\frac{\lambda \mathbf{a}_\lambda}{G(\mathbf{a}_\lambda(1-1/n))} \leq \frac{3}{4n^2}$$

whence $\lim_{\lambda\to 0}(\mathbf{m}_\lambda/\mathbf{n}_\lambda) = 0$ and $(2\mathbf{m}_\lambda + 1)G(\mathbf{a}_\lambda) < 1$.

Finally, fix $z \in (0\,,1)$ and consider n large enough, so that $1 - 1/n > z$. Then for $\lambda \in (0\,,\lambda_n)$, there holds $\varepsilon_\lambda \leq 1/n$, whence

$$G(\mathbf{a}_\lambda z)\mathbf{m}_\lambda \simeq \frac{G(\mathbf{a}_\lambda z)}{G(\mathbf{a}_\lambda(1-\varepsilon_\lambda))} \geq \frac{G(\mathbf{a}_\lambda z)}{G(\mathbf{a}_\lambda(1-1/n))} \to \infty$$

as $\lambda \to 0$, since $z < 1 - 1/n$, by $\mathbf{RV}(\infty)$. □

A.1. Examples

We finally compute in details the time scale $\mathbf{a}_\lambda$, recall (III.2.2), for various examples.

Example 1. The first example is the case where there is $a_0 > 0$ such that $\nu((-\infty\,,a_0)) = 0$ and $\nu([a_0\,,a_0+\varepsilon)) > 0$ for all $\varepsilon > 0$. First, since $G(t) \leq e^{-a_0 t}$, we have

$$1 = \lambda \int_0^{\mathbf{a}_\lambda} \frac{1}{G(s)}\,\mathrm{d}s \geq \frac{\lambda}{a_0}(e^{a_0\mathbf{a}_\lambda} - 1)$$

whence

$$\mathbf{a}_\lambda \leq \frac{1}{a_0}\log(1 + a_0/\lambda). \tag{A.6}$$

Conversely, for all $\varepsilon > 0$, there is $c_\varepsilon > 0$ such that $G(t) \geq c_\varepsilon e^{-(a_0+\varepsilon)t}$. Thus,

$$1 = \lambda \int_0^{\mathbf{a}_\lambda} \frac{1}{G(s)}\,\mathrm{d}s \leq \frac{\lambda}{c_\varepsilon(a_0+\varepsilon)}(e^{(a_0+\varepsilon)\mathbf{a}_\lambda} - 1)$$

whence

$$\mathbf{a}_\lambda \geq \frac{1}{a_0+\varepsilon}\log(1 + c_\varepsilon(a_0+\varepsilon)/\lambda). \tag{A.7}$$

Gathering (A.6) and (A.7), we easily deduce that

$$\mathbf{a}_\lambda \sim \frac{1}{a_0}\log(1/\lambda).$$

Example 2. Here we examine the example where $G(t) = e^{-t^\alpha}$ for all $t \geq 0$ and for some $\alpha \in (0\,,1)$. We prove that G satisfies $\mathbf{RV}(\infty)$ and that $\mathbf{a}_\lambda \sim [\log(1/\lambda)]^{1/\alpha}$.

We have
$$\frac{G(x)}{G(xt)} = \exp(-x^\alpha(1-t^\alpha)) \xrightarrow[x\to\infty]{} t^\infty.$$
Furthermore, setting $\phi(s) = 1/(G(s)s^\alpha)$, we have
$$\phi'(s) = \frac{\alpha s^{\alpha-1}e^{s^\alpha}s^\alpha - \alpha s^{\alpha-1}e^{s^\alpha}}{s^{2\alpha}} = \alpha\frac{\phi(s)}{s}(s^\alpha - 1).$$
We deduce that $\phi(s) + s\phi'(s) \underset{s\to\infty}{\sim} \alpha s^\alpha \phi(s)$, whence
$$x\phi(x) = \int_0^x [\phi(s) + s\phi'(s)]\,\mathrm{d}s \underset{x\to\infty}{\sim} \int_0^x \alpha s^\alpha \phi(s)\,\mathrm{d}s$$
i.e.
$$\int_0^x \frac{1}{G(s)}\,\mathrm{d}s \underset{x\to\infty}{\sim} \frac{x^{1-\alpha}}{\alpha}\frac{1}{G(x)}.$$
We deduce that
$$1 = \lambda\int_0^{\mathbf{a}_\lambda} \frac{1}{G(s)}\,\mathrm{d}s \sim \frac{\lambda}{\alpha}\frac{\mathbf{a}_\lambda^{1-\alpha}}{G(\mathbf{a}_\lambda)} = \frac{\lambda}{\alpha}\mathbf{a}_\lambda^{1-\alpha}e^{\mathbf{a}_\lambda^\alpha}.$$
It is not hard to conclude that
$$\mathbf{a}_\lambda \sim (\log(1/\lambda))^{1/\alpha}\,.$$

Example 3. We finally consider the law $\Gamma(\beta, 1)$, for $\beta > 0$, which has the density
$$f_\beta(x) = \frac{x^{\beta-1}e^{-x}}{\Gamma(\beta)}\mathbf{1}_{\{x>0\}}.$$
Its Laplace transform is given, for $t > 0$, by
$$G(t) = \int_{\mathbb{R}_+} \frac{x^{\beta-1}e^{-x}}{\Gamma(\beta)}e^{-xt}\,\mathrm{d}x = \frac{1}{(1+t)^\beta}$$
and satisfies $\mathbf{RV}(\beta)$ since, for all $t > 0$,
$$\frac{G(x)}{G(xt)} = \frac{(1+xt)^\beta}{(1+x)^\beta} \xrightarrow[x\to\infty]{} t^\beta.$$

We have
$$1 = \lambda\int_0^{\mathbf{a}_\lambda} \frac{1}{G(s)}\,\mathrm{d}s = \lambda\int_0^{\mathbf{a}_\lambda} (1+s)^\beta\,\mathrm{d}s = \frac{\lambda}{\beta+1}\Big((1+\mathbf{a}_\lambda)^{\beta+1} - 1\Big)$$
whence
$$\mathbf{a}_\lambda = \left(1 + \frac{\beta+1}{\lambda}\right)^{1/(\beta+1)} - 1 \underset{\lambda\to 0}{\sim} \left(\frac{\beta+1}{\lambda}\right)^{1/(\beta+1)}.$$

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